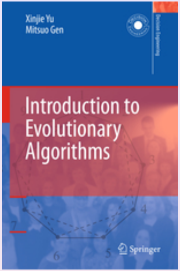
# Multi-Objective Optimization Using Metaheuristics

Carlos A. Coello Coello

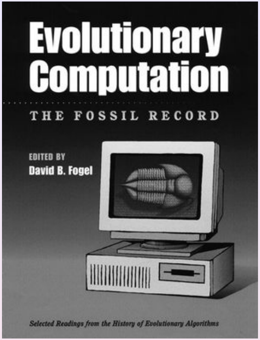
*ccoello@cs.cinvestav.mx*
CINVESTAV-IPN
Evolutionary Computation Group (EVOCINV)
Computer Science Department
Av. IPN No. 2508, Col. San Pedro Zacatenco
México, D.F. 07360, MEXICO

Eindhoven University of Technology
Eindhoven, The Netherlands July 2023
**LECTURE 2**

# Evolutionary Algorithms

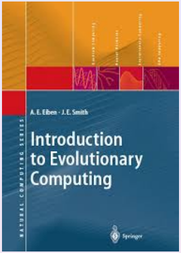



With no doubt, the most popular metaheuristic today are **evolutionary algorithms** (EAs), which are inspired on Darwin's survival of the fittest principle.




The idea of using techniques based on the emulation of the mechanism of natural selection (described in Darwin's evolutionary theory) to solve problems can be traced back to the early 1930s [Fogel, 1995].

# Evolutionary Algorithms




However, it was not until the 1960s that the three main techniques based on this notion were developed: genetic algorithms [Holland, 1962], evolution strategies [Schwefel, 1965] and evolutionary programming [Fogel, 1966]. These approaches, which are now collectively denominated "evolutionary algorithms," have been very effective for single-objective optimization.

The basic operation of an evolutionary algorithm (EA) is the following. First, they generate a set of possible solutions (called a “population”) to the problem at hand. Such a population is normally generated in a random manner. Each solution in the population (called an “individual”) encodes all the decision variables of the problem. In order to assess their suitability, a fitness function must be defined. Such a fitness function is a variation of the objective function of the problem that we wish to solve.

Then, a selection mechanism must be applied in order to decide which individuals will “mate.” This selection process is normally based on the fitness contribution of each individual (i.e., the fittest individuals have a higher probability of being selected). Upon mating, a set of “offspring” are generated. Such offspring are “mutated” (this operator produces a small random change, with a low probability, on the contents of an individual), and constitute the population to be evaluated at the following iteration (called a “generation”). This process is repeated until reaching a stopping condition (normally, a maximum number of generations).

The potential of evolutionary algorithms for solving multi-objective optimization problems dates back to the following PhD thesis:
Richard Rosenberg, **Simulation of genetic populations with biochemical properties**, PhD thesis, University of Michigan, Ann Arbor, Michigan, USA, June 1967.

# A Bit of History

Although Rosenberg's PhD thesis states, for the very first time, the possibility of using genetic algorithms to solve a multi-objective problem, no implementation is provided. The reason is that the bi-objective problem that he aimed to solve was transformed into a single-objective problem (the additional objective was transformed into a constraint).

Rosenberg's suggestion consisted in using multiple *properties* (closeness to some specific chemical composition) in his simulation of the genetics and chemistry of a population of single-cell organisms. Since his actual implementation adopted only one property, no multi-objective approach was required.

### VEGA

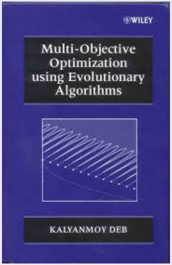
It was John David Schaffer the first in developing an actual implementation of a multi-objective evolutionary algorithm (MOEA), which is provided in his PhD thesis:
John David Schaffer, **Multiple Objective Optimization with Vector Evaluated Genetic Algorithms**, PhD thesis, Vanderbilt University, USA, 1984.

## VEGA

Schaffer's approach was called **Vector Evaluated Genetic Algorithm** (VEGA) and was published at the *First International Conference on Genetic Algorithms* in 1985.

J. David Schaffer, "**Multiple Objective Optimization with Vector Evaluated Genetic Algorithms**" , in *Genetic Algorithms and their Applications: Proceedings of the First International Conference on Genetic Algorithms*, pp. 93–100, Lawrence Erlbaum, New Jersey, USA, 1985.

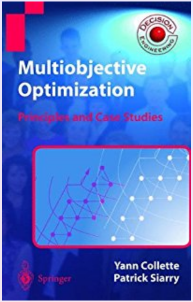
# A Taxonomy of MOEAs




## The Old Days

- Non-Elitist Non-Pareto-based Methods
  - Lexicographic Ordering
  - Linear Aggregating Functions
  - VEGA
  - $\varepsilon$-Constraint Method
  - Target Vector Approaches

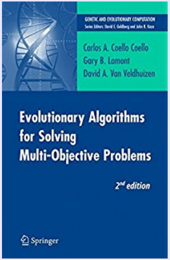
# A Taxonomy of MOEAs




## The Old Days

- Non-Elitist Pareto-based Methods
  - Pure Pareto ranking
  - MOGA
  - NSGA
  - NPGA and NPGA 2

# A Taxonomy of MOEAs




## Contemporary Approaches

- Elitist Pareto-based Methods
  - SPEA and SPEA2
  - NSGA-II
  - PAES, PESA and PESA II
  - Micro-genetic Algorithm for Multi-Objective Optimization and $\mu$GA$^2$
  - MOEAs that the world forgot

## Recent Approaches

- MOEA/D
- Indicator-Based Approaches
  - IBEA
  - SMS-EMOA
  - HyPE
  - Other Approaches
- NSGA-III

## Non-Elitist Non-Pareto-based Methods

Within this group, we will consider the oldest MOEAs reported in the literature, which do not adopt Pareto optimality in their selection mechanism and don't retain the nondominated solutions generated during the evolutionary process (i.e., they are non-elitist).

These MOEAs are simple and efficient, but they are also naive and ineffective (particularly for dealing with problems having more than 3 objectives).

# Non-Elitist Non-Pareto-based Methods

## Lexicographic Ordering

In this case, the objective functions have to be ranked based on their importance.

The original multi-objective optimization problem is re-stated as:

$$\text{Minimize} \quad f_1(\vec{x}) \tag{1}$$

subject to:

$$g_j(\vec{x}) \leq 0; \quad j = 1, 2, \ldots, m \tag{2}$$

and we obtain $\vec{x}_1^*$ and $f_1^* = f(\vec{x}_1^*)$.

## Lexicographic Ordering

Then, a second problem is stated as:

$$\text{Minimize} \quad f_2(\vec{x}) \tag{3}$$

subject to:

$$g_j(\vec{x}) \leq 0; \quad j = 1, 2, \ldots, m \tag{4}$$

$$f_1(\vec{x}) = f_1^* \tag{5}$$

and we obtain $\vec{x}_2^*$ and $f_2^* = f_2(\vec{x}_2^*)$.

# Non-Elitist Non-Pareto-based Methods

## Lexicographic Ordering

This procedure is repeated until all the objectives had been considered.

When used with evolutionary algorithms, some authors have randomly selected an objective at each generation.

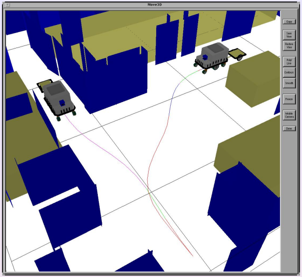
Other authors have adopted a scheme in which selection is performed by comparing only with respect to the most important objective. In this case, if there is a tie, the second most important objective is adopted in the comparison, and so on.

### Lexicographic Ordering

The **main advantage** of this approach is its simplicity, which involves a high computational efficiency.

It **main disadvantage** is that the performance of this approach depends on the ordering imposed on the objectives. Also, this approach is not suitable for problems having more than 2 objectives.

# Non-Elitist Non-Pareto-based Methods



## Lexicographic Ordering

Some applications of this approach are the following:

- Compaction of circuits using symbolic encoding [Fourman, 1985]
- Robot motion planning [Gacôgne, 1999]
- Airline crew rostering [El Moudani et al., 2001]

# Non-Elitist Non-Pareto-based Methods

## Linear Aggregating Functions

This was a very popular approach in the early days of evolutionary multi-objective optimization and some researchers still use it (e.g., in engineering and in Operations Research).

The core idea of this approach is quite simple: to transform a multi-objective problem into a scalar problem by performing a weighted sum of the objectives:

$$\min \sum_{i=1}^{k} w_i f_i(\vec{x}) \tag{6}$$

where $w_i \geq 0$ are the weights representing the relative importance of each of the $k$ objectives of the problem (the objectives need to be properly scaled). It is normally assumed that: $\sum_{i=1}^{k} w_i = 1$.

# Non-Elitist Non-Pareto-based Methods

## Linear Aggregating Functions

The **main advantages** of this approach are its simplicity and efficiency.

Its **main disadvantages** are: (1) the difficulty of defining a set of weights that allows the generation of most of the Pareto front, and (2) the fact that, regardless of the weights adopted, this approach is unable to generate non-convex portions of the Pareto front. See for example:
I. Das and J. Dennis, J. (1997), "**A Closer Look at Drawbacks of Minimizing Weighted Sums of Objectives for Pareto Set Generation in Multicriteria Optimization Problems**", *Structural Optimization*, **14**(1):63–69.

# Non-Elitist Non-Pareto-based Methods

## Linear Aggregating Functions

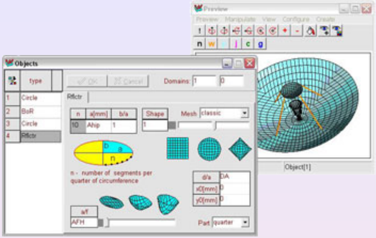
Yaochu Jin proposed in 2001 a clever scheme for generating weights in which the Pareto front is rotated. He showed that in this case, the use of a linear aggregating function is able to generate non-convex portions of the Pareto front. However, this approach cannot be easily generalized to problems with more than two objectives.

# Non-Elitist Non-Pareto-based Methods

## Linear Aggregating Functions

For many years, the use of a linear aggregating function within a MOEA has been seen as a bad idea. Nevertheless, there is solid evidence of the usefulness of this sort of approach in some classes of problems (e.g., in multi-objective combinatorial optimization).

# Non-Elitist Non-Pareto-based Methods



## Linear Aggregating Functions

Some applications of this approach are the following:

- Task planning [Jakob, 1992].
- Design of controllers [Donha, 1997].
- Design of optical filters for lamps [Eklund, 2001].
- Design of wire-antenna geometries [Van Veldhuizen et al., 1998].

# Non-Elitist Non-Pareto-based Methods

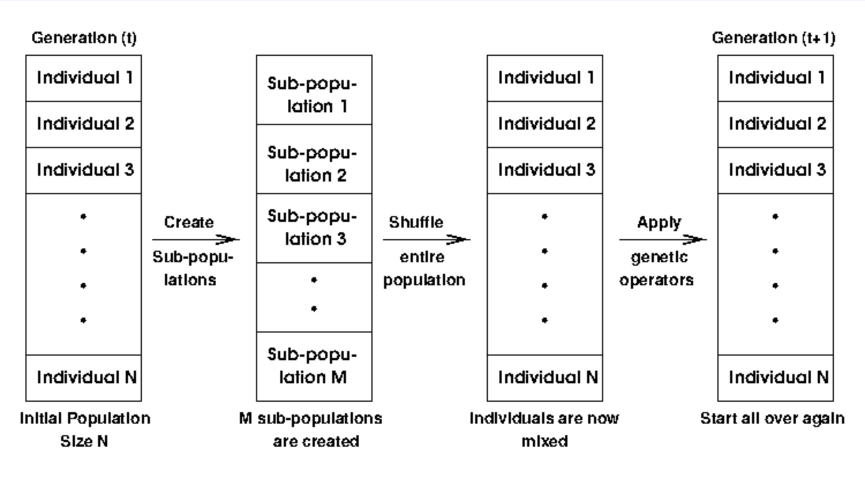
## VEGA

As indicated before, the **Vector Evaluated Genetic Algorithm** (VEGA) was proposed by David Schaffer in 1984, as part of his PhD thesis entitled “Multiple Objective Optimization with Vector Evaluated Genetic Algorithms”.

The description of VEGA was published in the proceedings of the *First International Conference in Genetic Algorithms* (ICGA’1985).

VEGA was originally applied to a machine learning problem.

# Non-Elitist Non-Pareto-based Methods

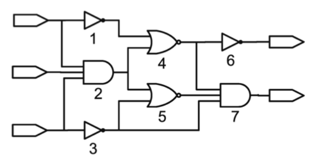


## VEGA

Its **main advantages** are: (1) it is easy to implement (only the selection mechanism of a simple genetic algorithm needs to be modified) and (2) it is efficient.

Its **main disadvantages** are: (1) if proportional selection is adopted, VEGA performs similarly to a linear aggregating function, and (2) it has a problem called “middling”, which was identified by Schaffer. Basically, VEGA’s selection mechanism not only omits the use of Pareto optimality, but opposes it. Also, it has not explicit mechanism to maintain diversity.

# Non-Elitist Non-Pareto-based Methods




## VEGA

Some applications of this approach are the following:

- Groundwater pollution containment [Ritzel, 1994].
- Optimum actuator selection [Rogers, 2000].
- Gate-level design of combinational logic circuits [Coello et al., 2000].
- Constraints-handling [Surry, 1997; Coello, 2000].

## $\varepsilon$-Constraint Method

In this case, we optimize first the objective function of our choice, and the others are considered as constraints bounded by certain allowable levels, which are called $\varepsilon_i$.

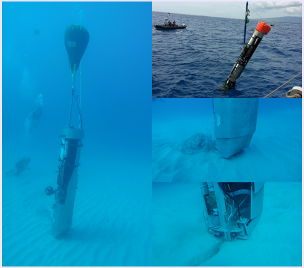
Then, we perform a single-objective optimization subject to the given constraints. After that, we modify the levels $\varepsilon_i$ and perform another single-objective optimization. This process is repeated several times. This will allow us to generate the whole Pareto optimal set even if the Pareto front is non-convex or disconnected. Evidently, in this case, an evolutionary algorithm is used for the single-objective optimizations.

## $\varepsilon$-Constraint Method

The **main advantage** of this approach is its relative simplicity and its generality (it works with any sort of multi-objective problem).

Its **main disadvantages** are: (1) it requires obtaining the Nadir point, which is something difficult for problems with more than 2 objectives, and (2) it is a computationally expensive approach, since many single-objective optimizations are normally required.

# Non-Elitist Non-Pareto-based Methods



## $\varepsilon$-Constraint Method

Some applications of this approach are the following:

- Preliminary design of marine vehicles [Lee, 1997].
- Groundwater pollution [Chetan, 2000].
- Fault tolerant design [Schott, 1995].
- Environmental engineering [Kumar, 2002].

# Non-Elitist Non-Pareto-based Methods

## Target Vector Approaches

Here, we consider approaches in which the user (DM) defines a set of goals (or targets) that aims to attain for each objective function. Then, the evolutionary algorithm tries to minimize the differences between the current solutions and this target of goals (different metrics can be adopted for this).

Strictly speaking, these approaches are also aggregating functions. However, they are considered as a separate type of technique, because they rely on nonlinear aggregating functions and, therefore, under certain conditions, they are able to generate non-convex portions of the Pareto front.

# Non-Elitist Non-Pareto-based Methods


## Target Vector Approaches

The most popular hybrids within this class are the following:

- EA + Goal Programming [Deb, 1999; Wienke, 1992; Sandgren, 1994]
- EA + Goal Attainment [Wilson, 1993; Zebulum, 1998]
- EA + Min-Max Optimum [Hajela, 1992; Coello, 1998]

# Non-Elitist Non-Pareto-based Methods

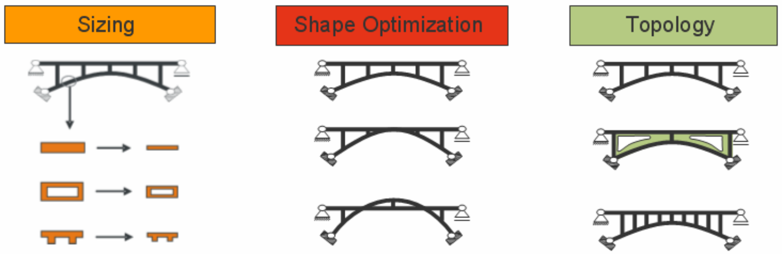
### Target Vector Approaches

Their **main advantages** are their computational efficiency and simplicity.

Their **main disadvantages** are related to the difficulty for generating the goals that we aim to achieve. Also, some of these approaches can behave (under certain conditions) in an ambiguous form.

Additionally, some of these approaches require that the goals are defined in the feasible region in order to guarantee that the solutions generated are Pareto optimal.

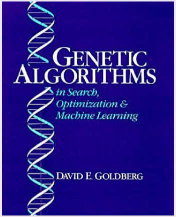
# Non-Elitist Non-Pareto-based Methods




## Target Vector Approaches

Some applications of these approaches are the following:

- Design of IIR digital filters [Wilson, 1993].
- Structural optimization [Sandgren, 1994; Hajela, 1992]
- Counterweight balancing of robot arms [Coello, 1998].




## Origins

David Goldberg criticized VEGA in his famous book on genetic algorithms [Goldberg, 1989] and then proposed the idea of using a fitness assignment scheme based on Pareto optimality for a MOEA.

# Non-Elitist Pareto-based Methods

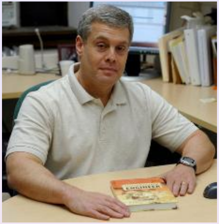
## Origins

Goldberg proposed to identify the nondominated solutions in the population. These individuals would be assigned a better fitness than any dominated solution. Then, these nondominated solutions would be temporarily removed from the population so that a new ranking could be performed. The process would be repeated until the whole population had received a fitness value. This algorithm is known today as **nondominated sorting** and this process is normally known as **Pareto ranking.**

## Origins

Goldberg also indicated that, because of stochastic noise, a multi-objective genetic algorithm would eventually converge to a single solution. Thus, he suggested to block the selection process in order to maintain diversity. This would allow the generation of several elements of the Pareto optimal set in a single run. This mechanism is now included in most MOEAs and is called **density estimator.**

# Non-Elitist Pareto-based Methods



## Origins

Goldberg proposed the use of **fitness sharing**, which is an approach that he originally introduced in 1987. The idea is to subdivide the population into several subpopulations based on the similarity of their individuals. Similarity can be measured in phenotypic (i.e., decoded) space or in genotypic (i.e., binary) space.

# Non-Elitist Pareto-based Methods

## Origins

The following equation is adopted:

$$\phi(d_{ij}) = \begin{cases} 1 - \left(\frac{d_{ij}}{\sigma_{sh}}\right)^{\alpha}, & d_{ij} < \sigma_{share} \\ 0, & \text{otherwise} \end{cases} \tag{7}$$

Here: $\alpha = 1$, $d_{ij}$ is a metric that indicates the distance between solutions $i$ and $j$, and $\sigma_{share}$ is a user-defined parameter (also called **niche radius**).

The fitness of an individual $i$ is then modified using:

$$f_{s_i} = \frac{f_i}{\sum_{j=1}^{M} \phi(d_{ij})} \tag{8}$$

where $M$ is the number of individuals that are located in the neighborhood of the $i^{th}$ individual. So, individuals are penalized as more individuals are sharing the same niche. Ideally, every individual should be alone in its own niche.

# Non-Elitist Pareto-based Methods

### Pure Pareto Ranking

This refers to the use of Pareto ranking without the density estimator. This sort of approach was rare even in the old days, since the density estimator is important to diversify the search.

The main use of this approach was in applications in which the authors were really interested in finding a single (or a few) nondominated solution(s) rather than the whole Pareto front. This sort of focus is more common in engineering applications.

# Non-Elitist Pareto-based Methods

## Pure Pareto Ranking

Some applications of this approach are the following:

- Groundwater monitoring problems [Cieniawski, 1995].
- Pump scheduling in water supply [Schwab, 1996;Savic, 1997].
- Preliminary design of submarines [Thomas, 1998].
- Planning of power distribution systems [Ramírez, 1999].

# Non-Elitist Pareto-based Methods

## MOGA

It turns out that Goldberg's idea can be improved and this is precisely what Carlos M. Fonseca did in the so-called **Multi-Objective Genetic Algorithm** (MOGA), which he proposed (together with Peter Fleming) in 1993.

See:
Carlos M. Fonseca and Peter J. Fleming, "**Genetic Algorithms for Multiobjective Optimization: Formulation, Discussion and Generalization**, in Stephanie Forrest (Ed), *Proceedings of the Fifth International Conference on Genetic Algorithms*, pp. 416–423, Morgan Kauffman Publishers, San Mateo, California, 1993.

# Non-Elitist Pareto-based Methods

## MOGA

In MOGA, the rank of an individual is given by:

$$\text{rank}(x_i, t) = 1 + p_i^{(t)} \tag{9}$$

Every nondominated individual is assigned a rank of 1, and dominated individuals are penalized based on the number of solutions that dominate it.

Fitness is assigned using the following procedure:

- Sort the population based on their rank.
- Assign fitness to each individual by interpolating from the best rank (1) to the worst ($n \leq M$), where $M$ is the population size. Interpolation is usually, but not necessarily, linear.
- Average the fitness values of the individuals with the same rank, so that all of them are sampled in the same way.
- MOGA uses fitness sharing (but Fonseca provides a procedure to compute $\sigma_{share}$) and mating constraints.

## MOGA

MOGA became popular in the mid-1990s because it was not only relatively efficient ($O(N^2)$), but also quite effective. Additionally, it was implemented in MatLab, which motivated to several researchers in automatic control to use it.

Some comparative studies from the mid-1990s showed that MOGA was the best MOEA from the first generation.

The implementation of MOGA was not completely straightforward, since it contained lots of details. In fact, Fonseca claims that he included elitism in his implementation, but this mechanism is not described in his paper from 1993.

## MOGA

Some applications of this approach are the following:

- Co-synthesis of hierarchical heterogeneous distributed embedded systems [Dick, 1998].
- Optimization of distributed active magnetic bearing controllers [Schroder, 1997].
- Process fault diagnosis [Marcu, 1999].
- Plane truss optimization [Narayanan, 1999].
- Forest management [Ducheyne, 2001].
- Design of gas turbines [Fonseca, 1995].

## NSGA

The **Nondominated Sorting Genetic Algorithm** (NSGA) was proposed by Srinivas and Deb in 1994. It was the first MOEA published in a specialized journal (*Evolutionary Computation*).

NSGA follows very closely Goldberg's informal description of the Pareto ranking process that a MOEA should adopt.

# Non-Elitist Pareto-based Methods


## NSGA

NSGA was published in:
N. Srinivas and Kalyanmoy Deb, **Multiobjective Optimization Using Nondominated Sorting in Genetic Algorithms**, *Evolutionary Computation*, **2**(3):221-248, Fall 1994.

# Non-Elitist Pareto-based Methods

## NSGA

NSGA identifies first the individuals which are nondominated with respect to the whole population. Such individuals are assigned a “dummy” fitness value and are then removed from the population. The process is repeated with the remainder of the population, assigning dummy fitness values that decrease with each further layer (such that the first layer has the highest values). This process finishes once all the individuals have been assigned a fitness value.

The density estimator is also fitness sharing (using the dummy fitness value assigned by the nondominated sorting procedure). However, in this case, similarity is measured in decision variable space (unlike MOGA, which measures it in objective function space).

# Non-Elitist Pareto-based Methods

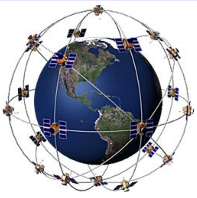
## NSGA

The nondominated sorting algorithm adopted by NSGA is $O(N^3)$, which contrasts with the (one-pass) Pareto ranking procedure of MOGA, which is $O(N^2)$.

The few comparative studies of MOEAs performed in the later 1990s indicated that NSGA was slow and did not produce very good results. Also, it was highly sensitive to the value of $\sigma_{share}$.

MOGA is discussed in NSGA's journal paper. However, results are only compared with respect to VEGA. Most comparative studies from the late 1990s indicated that MOGA consistently outperformed NSGA.

# Non-Elitist Pareto-based Methods



## NSGA

Some applications of this approach are the following:

- Long term groundwater monitoring [Reed et al., 2001].
- Optimization of fuzzy logic scheduled controllers for missile autopilot design [Blumel, 2001]
- Satellite constellation design [Mason, 1999].
- Multi-objective optimization in computational fluid dynamics [Marco, 1999].

## NPGA

The *Niched-Pareto Genetic Algorithm* (NPGA) was proposed by Jeffrey Horn in a Technical Report from 1993 and it was published at an international conference in 1994.

It was published in:
Jeffrey Horn, Nicholas Nafpliotis and David E. Goldberg, "**A Niched Pareto Genetic Algorithm for Multiobjective Optimization**", in *Proceedings of the First IEEE Conference on Evolutionary Computation*, Vol. 1, pp. 82–87, IEEE Press, Piscataway, New Jersey, USA, June 1994.

## NPGA

NPGA adopts a variant of binary tournament selection in which two (randomly selected individuals) compete in terms of Pareto dominance. Each of these two individuals is compared with respect to a sample of the population whose size is a user-defined parameter (normally, 10% of the total population size is adopted). So, the tournament only has two possible outcomes:

1. One of the two individuals is nondominated and the other one is dominated. In this case, the nondominated individual wins the tournament and, therefore, it is selected.
2. There is a tie (either both are nondominated or both are dominated). In this case, fitness sharing is applied to the two competing individuals. The technique adopted is called *equivalence class sharing* and is applied both in decision variable space and objective function space.

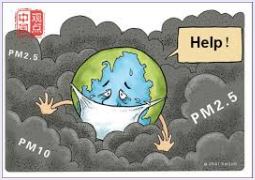
## NPGA

Horn showed that a small sample from the total population was sufficient to estimate Pareto optimality of an individual and produced the fastest MOEA of its generation.

This is the only MOEA in which David Goldberg appears as a co-author (Goldberg was Horn's PhD advisor at the University of Illinois at Urbana-Champaign).

The few comparative studies from the late 1990s indicated that NPGA was better than NSGA, but no better than MOGA. In fact, if the tournament size is made equal to the population size, NPGA would become MOGA.

# Non-Elitist Pareto-based Methods




## NPGA

Some applications of this approach are the following:

- X-ray plasma spectroscopy [Golovkin, 2000].
- Feature selection [Emmanouilidis, 2000].
- Fault tolerant design [Schott, 1995].
- Reduction of traffic generated urban air and noise pollution [Haastrup & Pereira, 1997].

# Non-Elitist Pareto-based Methods

## NPGA 2

Erickson et al. [2001] proposed a variant of NPGA called NPGA 2: Mark Erickson, Alex Mayer and Jeffrey Horn, "**The Niched Pareto Genetic Algorithm 2 Applied to the Design of Groundwater Remediation Systems**", in Eckart Zitzler, Kalyanmoy Deb, Lothar Thiele, Carlos A. Coello Coello and David Corne (Eds), *First International Conference on Evolutionary Multi-Criterion Optimization*, pp. 681–695, Springer-Verlag, Lecture Notes in Computer Science No. 1993, 2001.

## NPGA 2

NPGA 2 adopts Pareto ranking using the original binary tournament selection from NPGA. However, a new fitness sharing scheme is adopted in this case, in which the niche count is computed using individuals from the next (partially filled) generation, instead of using the individuals from the current generation. This scheme is called **continuously updated fitness sharing** [Oei, 1991].

NPGA 2 has apparently being used only in the design of groundwater remediation systems [Erickson, 2001].

## What is Elitism?

In single-objective optimization, **elitism** is an operator by which the best solution in the population passes intact to the next generation (i.e., it is not affected by crossover or mutation).

In the context of multi-objective optimization, elitism operates in a similar way, but in this case, we need to retain (all) the nondominated solutions generated by a MOEA. Since it is impractical to retain all of these solutions, it is normally the case, that some sort of bound is set on the maximum number of solutions that are retained. This is particularly important in MOEAs in which the elitist solutions play a role in the selection mechanism (e.g., SPEA).

## Forms of Elitism

The two main forms in which elitism is normally implemented are:

1. Through the use of an **external archive** (also called **external population**), which is a data structure that resides in main memory and which stores the nondominated solutions generated during the evolutionary process
2. Using a **plus selection mechanism** in which the population of parents is merged with the population of offspring and only the best half is retained.

# Elitist Pareto-based Methods

## Why is elitism important?

Elitism is required to guarantee convergence of a MOEA to the true Pareto optimal set of a multi-objective optimization problem, as proved by Rudolph and Agapie [2001].

See:
Günter Rudolph and Alexandru Agapie, "**Convergence Properties of Some Multi-Objective Evolutionary Algorithms**", in *Proceedings of the 2000 IEEE Conference on Evolutionary Computation*, Vol. 2, pp. 1010–1016, IEEE Press, Piscataway, New Jersey, USA July 2000.

# Elitist Pareto-based Methods

## SPEA

Eckart Zitzler [1998,1999] proposed in his PhD thesis the **Strength Pareto Evolutionary Algorithm** (SPEA) as a MOEA that integrates different mechanisms from previous approaches.

## SPEA

See:
Eckart Zitzler and Lothar Thiele, "**Multiobjective Evolutionary Algorithms: A Comparative Case Study and the Strength Pareto Approach**", *IEEE Transactions on Evolutionary Computation*, **3**(4):257-271, November 1999.

SPEA adopts an external archive and somehow generalized this notion of elitism within MOEAs. At each generation, the nondominated solutions from the population are copied to this archive, and the archive participates in the selection process. For each individual in the external archive, a "strength" value is computed. This value is similar to the rank in MOGA, since it is proportional to the number of solutions that a certain individual dominates.

## SPEA

The fitness of each individual in the population is computed based on the strengths of all the individuals in the external archive to which a certain individual dominates.

Zitzler realized that if the size of the archive was not bounded, the selection pressure would dilute as the number of nondominated solutions grew very quickly. Thus, he decided to prune the archive using a clustering technique called **average linking method** [Morse, 1989], once a certain (pre-defined) limit was reached.

## SPEA

Some applications of this approach are the following:

- Exploration of software schedules for digital signal processors [Zitzler, 1999].
- Planning of medical treatments [Petrovski, 2001].
- Dose optimization problems in brachytherapy [Lahanas, 2001].
- Non-invasive atrial disease diagnosis [de Toro, 2003].
- Rehabilitation of a water distribution system [Cheung, 2003].

### SPEA2

A revised (and improved) version of SPEA (called SPEA2) was proposed by Eckart Zitzler and his colleagues in 2001.

See:
Eckart Zitzler, Marco Laumanns and Lothar Thiele, "**SPEA2: Improving the Strength Pareto Evolutionary Algorithm**, in K. Giannakoglou, D. Tsahalis, J. Periaux, P. Papailou and T. Fogarty (eds.), *EUROGEN 2001, Evolutionary Methods for Design, Optimization and Control with Applications to Industrial Problems*, pp. 95–100, Athens, Greece, 2002.

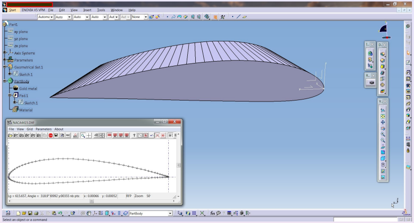
# Non-Elitist Pareto-based Methods

## SPEA2

SPEA2 has three main differences with respect to the original SPEA:

- It incorporates a fine-grain fitness assignment strategy which takes into consideration both the number of individuals that a solution dominates and the number of solutions by which it is dominated.
- A more efficient density estimator (a better clustering algorithm).
- A mechanism to truncate the external archive, which guarantees that bound solutions are retained.

# Elitist Pareto-based Methods



## SPEA2

Some applications of this approach are the following:

- Reduction of bloat in genetic programming [Bleuler, 2001].
- Airfoil design [Willmes, 2003].
- Portfolio optimization [Garcia, 2011].
- Optimization of diesel engine emissions and fuel economy [Hiroyasu, 2005].

## NSGA-II

The *Nondominated Sorting Genetic Algorithm II* (NSGA-II) was originally proposed by Kalyanmoy Deb and his students in 2000. However, most people only know the journal version of this paper, which appeared in 2002: Kalyanmoy Deb, Amrit Pratap, Sameer Agarwal and T. Meyarivan, "**A Fast and Elitist Multiobjective Genetic Algorithm: NSGA–II**, *IEEE Transactions on Evolutionary Computation*, Vol. 6, No. 2, pp. 182–197, April 2002.

## NSGA-II

NSGA-II is actually quite different from the original NSGA. It still adopts nondominated sorting, but only in a single pass (as MOGA). Also, it adopts a plus selection mechanism by which the parents population is merged with the offspring population, such that only the best half survives (this is an implicitly elitist scheme).

A key element of NSGA-II is its density estimator, which is called **crowded comparison operator**. This approach requires that solutions are sorted with respect to one objective. Then, each individual uses its previous and further neighbors to build a rectangle. When comparing two solutions, if there is a tie (i.e., either both are nondominated or both are dominated), the one with the larger perimeter wins (i.e., preference is given to solutions in more isolated regions of objective function space). This density estimator requires no extra parameters and is quite efficient.

**NSGA-II**

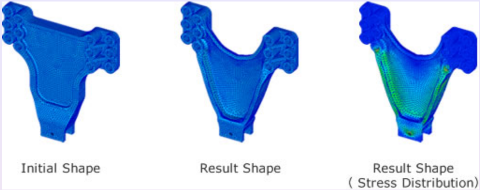
The elegance, effectiveness and efficiency of NSGA-II made it a standard in evolutionary multi-objective optimization for more than 10 years.

The fact that its source code has been made available also contributed to its popularity (it is probably the most popular MOEA ever).

### NSGA-II

However, NSGA-II does not work properly with more than 3 objectives, mainly because of its density estimator, which was conceived only for two objectives.

Additionally, there is experimental evidence that indicates that NSGA-II works better with real-numbers encoding than with binary encoding.




## NSGA-II

Some applications of this approach are the following:

- Shape optimization [Deb, 2001].
- Safety systems optimum design [Greiner, 2003].
- Optimization of processing conditions for polymer twin-screw extrusion (Gaspar-Cunha, 2002).
- Watershed water quality management [Dorn, 2003].
- Intensity modulated beam radiation therapy dose optimization [Lahanas, 2003].

# Elitist Pareto-based Methods

## PAES

The *Pareto Archived Evolution Strategy* (PAES) was proposed in 1999, but its journal version appeared in 2000.

See:
Joshua D. Knowles and David W. Corne, "**Approximating the Nondominated Front Using the Pareto Archived Evolution Strategy**", *Evolutionary Computation*, Vol. 8, No. 2, pp. 149–172, 2000.

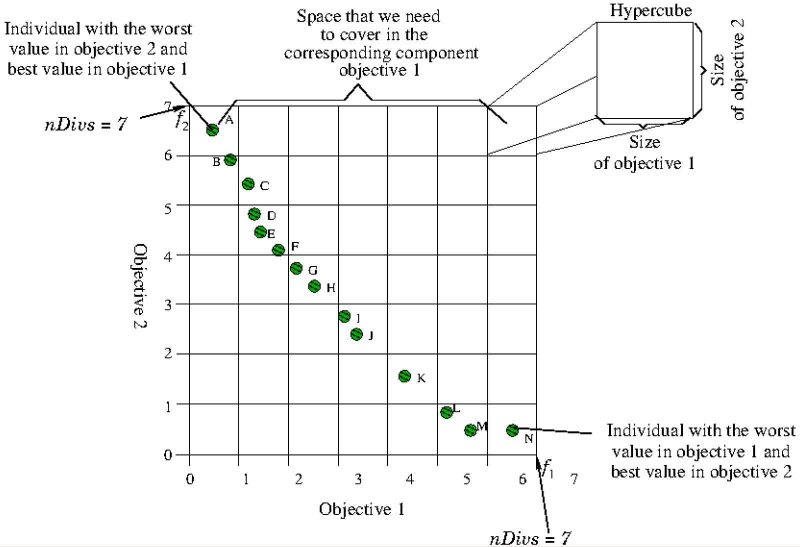
# Elitist Pareto-based Methods

## PAES

Conceptually speaking, PAES is perhaps the most simple MOEA that one can possibly design. It consists of a (1+1)-ES (i.e., a single parent which is mutated to produce an offspring). If the offspring dominates its parent, it is stored in an external archive and it becomes the parent in the next iteration.

The most interesting aspect of this approach is its external archive, which adopts a density estimator called *adaptive grid*. This density estimator only requires one parameter: the number of sub-divisions to be adopted in objective function space. Its main problem is that it was conceived only for two objectives and its generalization to any number of objectives doesn't seem possible.

# Elitist Pareto-based Methods



# Elitist Pareto-based Methods

## PAES

Some applications of this approach are the following:

- Telecommunications problems [Knowles, 1999].
- The adaptive distributed database management problem [Knowles, 2000].
- Flexible job shop scheduling [Rabiee, 2012].

## PESA

The *Pareto Envelope-based Selection Algorithm* (PESA) was proposed by David Corne in 2001.

See:
David W. Corne, Joshua D. Knowles and Martin J. Oates, "**The Pareto Envelope-based Selection Algorithm for Multiobjective Optimization**", in Marc Schoenauer et al. (editors), *Proceedings of the Parallel Problem Solving from Nature VI Conference*, pp. 839–848. Springer, 2000.

# Elitist Pareto-based Methods

### PESA

PESA uses a small internal population and a larger external population (similar to the one adopted by PAES).

Like PAES, PESA implicitly maintains a hyper-grid division of objective function space which allows it to keep track of the degree of crowding in different regions of the archive. However, in this case, and unlike PAES, the selection mechanism is based on this crowding measure. This same crowding measure is also used to decide which solutions will enter the external archive.

# Elitist Pareto-based Methods

## PESA-II

The *Pareto Envelope-based Selection Algorithm-II* (PESA II) was proposed by David Corne in 2001.

See:
David W. Corne, Nick R. Jerram, Joshua D. Knowles and Martin J. Oates, "**PESA-II: Region-based Selection in Evolutionary Multiobjective Optimization**", in Lee Spector et al. (editors), *Proceedings of the 2001 Genetic and Evolutionary Computation Conference (GECCO'2001)*, pp. 283–290, Morgan Kaufmann Publishers, San Francisco, California, July 2001.

## PESA-II

The only difference of PESA-II with respect to PESA is that, in this case, a region-based selection scheme is adopted.

In PESA-II, instead of assigning a selective fitness to an individual, selective fitness is assigned to the hyperboxes in objective space which are currently occupied by at least one individual in the current approximation to the Pareto front. Thus, a hyperbox is selected, and then one individual is randomly selected from this hyperbox. The authors of PESA-II argue that this scheme produces a better spread of solutions than the traditional individual-based selection scheme of PESA.

Since the adaptive grid can't be scaled, the selection schemes of both PESA and PESA-II cannot be used with more than 2 objectives.

PESA and PESA-II were used to solve some telecommunications problems [Corne, 2000; Corne, 2001].

## The Micro-Genetic Algorithm for Multi-Objective Optimization

It was proposed by Coello and Toscano [2001]. A micro-genetic algorithm has a very small population size (no more than 5 individuals) and adopts a reinitialization process to maintain diversity.

See:
Carlos A. Coello Coello and Gregorio Toscano Pulido, "**Multiobjective Optimization using a Micro-Genetic Algorithm**, in Lee Spector et al. (editors), *Proceedings of the 2001 Genetic and Evolutionary Computation Conference (GECCO'2001)*, pp. 274–282, Morgan Kaufmann Publishers, San Francisco, California, July 2001.

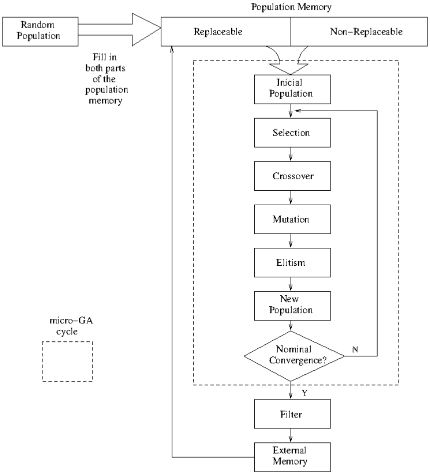
### The Micro-Genetic Algorithm for Multi-Objective Optimization

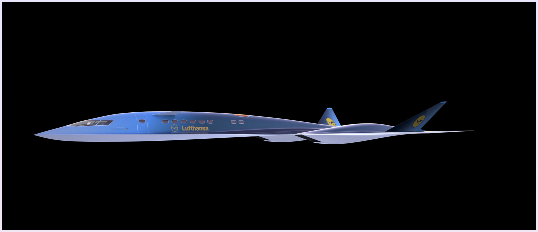
This approach adopts 3 forms of elitism and the adaptive grid of PAES (with a small modification that makes its more efficient.

This was a very fast MOEA. In the comparative studies performed in 2001, it was up to one order of magnitude faster than NSGA-II and produced solutions of similar quality.

The main drawback of this approach was that it required a high number of parameters (eight, from which at least three play a key role in its performance).

# Elitist Pareto-based Methods






## The Micro-Genetic Algorithm for Multi-Objective Optimization

Some applications of this approach are the following:

- Design of supersonic business jets [Chung et al., 2003].
- Structural optimization [Coello, 2002].
- Partitions of hardware/software systems [Fornaciari, 2003].
- Location of automatic voltage regulators in a radial distribution network [Mendoza, 2007].

## The Micro-Genetic Algorithm for Multi-Objective Optimization 2

This approach (called $\mu$GA$^2$, for short) was introduced by Toscano and Coello in 2003. This is the only fully self-adaptive MOEA that has been proposed so far.

See:
Gregorio Toscano Pulido and Carlos A. Coello Coello, "**The Micro Genetic Algorithm 2: Towards Online Adaptation in Evolutionary Multiobjective Optimization**", in Carlos M. Fonseca et al. (editors), *Evolutionary Multi-Criterion Optimization. Second International Conference, EMO 2003*, pp. 252–266, Springer. Lecture Notes in Computer Science. Volume 2632, Faro, Portugal, April 2003.

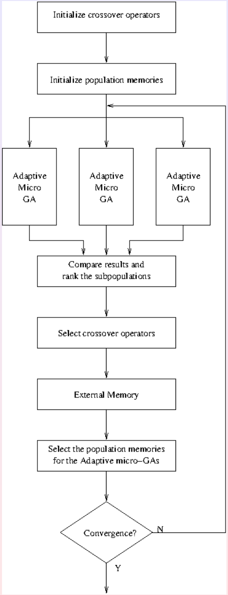
# Elitist Pareto-based Methods

## The Micro-Genetic Algorithm for Multi-Objective Optimization 2

The main motivation of the $\mu GA^2$ was to eliminate the 8 parameters required by the original algorithm. The $\mu GA^2$ uses on-line adaption mechanisms that make unnecessary the fine-tuning of any of its parameters.

The $\mu GA^2$ can decide when to stop (no maximum number of generations has to be provided by the user). The only parameter that it requires is the size of external archive (although there is obviously a default value for this parameter).

It has been used for solving reconfiguration problems considering power losses and reliability indices for medium voltage distribution network [Mendoza, 2009].


Initialize crossover operators
Initialize population memories
Adaptive Micro GA
Adaptive Micro GA
Adaptive Micro GA
Compare results and rank the subpopulations
Select crossover operators
External Memory
Select the population memories for the Adaptive micro–GAs
Convergence?
N
Y

## Incrementing Multi-Objective Evolutionary Algorithm (IMOEA)

Proposed by Tan et al. [2001]. It uses a dynamic population size, based on the current approximation of the Pareto front. It also adopts an adaptive niching method.

See:
K.C. Tan, T.H. Lee and E.F. Khor, "**Evolutionary Algorithms with Dynamic Population Size and Local Exploration for Multiobjective Optimization**", *IEEE Transactions on Evolutionary Computation*, Vol. 5, No. 6, pp. 565–588, December 2001.

# MOEAs that the World Forgot

## Constraint Method-Based Evolutionary Algorithm (CMEA)

Proposed by Ranjithan et al. [2001]. It is based on the $\epsilon$-constraint method.

See:
S. Ranji Ranjithan, S. Kishan Chetan and Harish K. Dakshima, "**Constraint Method-Based Evolutionary Algorithm (CMEA) for Multiobjective Optimization**", in Eckart Zitzler et al. (Eds), *First International Conference on Evolutionary Multi-Criterion Optimization*, pp. 299–313. Springer-Verlag. Lecture Notes in Computer Science No. 1993, 2001.

## Orthogonal Multi-Objective Evolutionary Algorithm (OMOEA)

Proposed by Zeng et al. [2004], this approach is based on orthogonal design and other statistical techniques. It adopts niching.

See:
Sanyou Y. Zeng, Lishan S. Kang and Lixin X. Ding, "**An Orthogonal Multi-objective Evolutionary Algorithm for Multi-objective Optimization Problems with Constraints**, *Evolutionary Computation*, Vol. 12, No. 1, pp. 77–98, Spring 2004.

## The **MaxiMin** Method

Originally proposed by Balling [2000, 2001], it uses an expression similar to compromise programming to assign fitness and estimate density with a single mathematical expression. This approach has been recently studied and improved [Menchaca, 2016].

See:
Richard Balling and Scott Wilson, "**The Maximim Fitness Function for Multi-objective Evolutionary Computation: Application to City Planning**", in Lee Spector et al. (editors), *Proceedings of the Genetic and Evolutionary Computation Conference (GECCO'2001)*, pp. 1079–1084, Morgan Kaufmann Publishers, San Francisco, California, July 2001.

## The Non-Generational GA for Multiobjective Optimization

It was proposed by Valuenzuela-Rendón and Uresti-Charre [1997]. It uses an aggregation function that combines dominance count with niching (it uses fitness sharing) in a steady-state genetic algorithm which was inspired on Learning Classifier Systems. This approach was extended in Borges and Barbosa [2000]. In this case, for each element in the population, a domination count is defined together with a neighborhood density measure based on a sharing function. Those two meawes are then non-linearly combined in order to define the individual's fitness.


- Manuel Valenzuela-Rendón and Eduardo Uresti-Charre, "**A Non-Generational Genetic Algorithm for Multiobjective Optimization**", in Thomas Bäck (Ed), *Proceedings of the Seventh International Conference on Genetic Algorithms*, pp. 658–665, Morgan Kaufmann Publishers, July 1997.
- Carlos C.H. Borges and Helio J.C. Barbosa, "**A Non-generational Genetic Algorithm for Multiobjective Optimization**", in *2000 IEEE Congress on Evolutionary Computation*, Vol. 1, pp. 172–179, IEEE Service Center, July 2000.

# MOEAs that the World Forgot

## The Method based on Distances and the Contact Theorem

It was proposed by Osyczka and Kundu [1996]. It uses a nonlinear aggregating function that estimates distances with respect to the ideal vector.

Andrzej Osyczka and Sourav Kundu. "**A modified distance method for multicriteria optimization using genetic algorithms**", *Computers in Industrial Engineering*, **30**(4):871-882, 1996.

## The Nash Genetic Algorithm

It was proposed by Sefrioui and Periaux [1996]. It uses a co-evolutionary scheme to try to approximate a Nash equilibrium point (in a Nash strategy, each player tries to optimize his/her own criterion, assuming that the other players' criteria are fixed). It uses a distance-based mutation operator. It requires certain mathematical calculations to define the model to be optimized with a genetic algorithm and its outcome is a single solution.

M. Sefrioui and J. Periaux. "**Nash Genetic Algorithms: examples and applications**", in *2000 IEEE Congress on Evolutionary Computation*, Vol. 1, pp. 509–516, IEEE Press, San Diego, California, July 2000.

## The Thermodynamic Genetic Algorithm

It was proposed by Kita [1996]. This is a multi-objective version of an algorithm originally proposed for combinatorial optimization. It incorporates the concept of entropy and the use of a cooling schedule in its selection mechanism.

Hajime Kita, Yasuyuki Yabumoto, Naoki Mori and Yoshikazu Nishikawa, "**Multi-Objective Optimization by Means of the Thermodynamical Genetic Algorithm**, in Hans-Michael Voigt et al. (Eds), *Parallel Problem Solving from Nature–PPSN IV*, Springer, Lecture Notes in Computer Science, pp. 504–512, Berlin, Germany, September 1996.

## The $\varepsilon$-MOEA

It was proposed by Deb [2003, 2005]. It is based on a relaxed form of Pareto dominance called $\varepsilon$-dominance [Laumanns et al., 2002]. This approach uses steady state selection and adopts an external population that incorporates $\varepsilon$-dominance.

Kalyanmoy Deb, Manikanth Mohan and Shikhar Mishra, "**Evaluating the epsilon-Domination Based Multi-Objective Evolutionary Algorithm for a Quick Computation of Pareto-Optimal Solutions**", *Evolutionary Computation*, Vol. 13, No. 4, pp. 501–525, Winter 2005.

# MOEAs that the World Forgot

## Use of Genders

Allenson (1992) used a variation of VEGA in which genders were used to distinguish between the two objective functions related to the planning of a path composed by several rectilinear pipe segments. In this approach, recombination is only possible between pairs of individuals having a different gender (a male and a female) and the gender is randomly assigned to an offspring. In the initial population, it is ensured that half of the population are male and half are female, but this balance is no longer maintained upon the application of the genetic operators. At each generation, the worst individual (chosen from one of the two genders) is eliminated and its place is taken by another individual (randomly selected) from its same gender.

Allenson used evolution strategies to implement the sexual attractors that modify the way in which recombination takes place.

Robin Allenson, "**Genetic algorithms with gender for multi-function optimisation**", Technical Report EPCC-SS92-01, Edinburgh Parallel Computing Centre, Edinburgh, Scotland, 1992.

## Use of Genders

Allenson's core idea was to model the sexual attraction that occurs in nature and which determines a not so random mating. In 1996, Lis and Eiben proposed a generalization of this approach in which there are as many genders as objectives.

Joanna Lis and A.E. Eiben, "**A Multi-Sexual Genetic Algorithm for Multiobjective Optimization**", in Toshio Fukuda and Takeshi Furuhashi (Eds), *Proceedings of the 1996 International Conference on Evolutionary Computation*, IEEE Press, pp. 59–64, Nagoya, Japan, 1996.

## MOEA/D

The *Multi-Objective Evolutionary Algorithm based on Decomposition* (MOEA/D) proposed by Zhang and Li [2007] is one of the most competitive MOEAs in current use. This approach decomposes a multi-objective problem into several single-objective optimization problems, which are simultaneously solved. Each subproblem is optimized using information from its neighboring subproblems, in contrast with similar approaches (e.g., MOGLS [Ishibuchi & Murata, 1996]). This MOEA is inspired on a mathematical programming technique called *Normal Boundary Intersection* (NBI) [Das, 1998].

Qingfu Zhang and Hui Li, "**MOEA/D: A Multiobjective Evolutionary Algorithm Based on Decomposition**", *IEEE Transactions on Evolutionary Computation*, Vol. 11, No. 6, pp. 712–731, December 2007.

## Indicator-based Selection

Perhaps the most important trend on the design of moderns MOEAs is the use of a performance measure in their selection mechanism.

## ESP

The **Evolution Strategy with Probability Mutation** uses a measure based on the hypervolume, which is scale independent and doesn't require any parameters, in order to truncate the contents of the external archive [Huband et al., 2003].

Simon Huband, Phil Hingston, Lyndon White and Luigi Barone, "**An Evolution Strategy with Probabilistic Mutation for Multi-Objective Optimisation**", in *Proceedings of the 2003 Congress on Evolutionary Computation (CEC'2003)*, Vol. 3, pp. 2284–2291, IEEE Press, Canberra, Australia, December 2003.

## IBEA

The **Indicator-Based Evolutionary Algorithm** is an algorithmic framework that allows the incorporation of any performance indicator in the selection mechanism of a MOEA [Zitzler et al., 2004]. It was originally tested using the hypervolume and the binary $\epsilon$ indicator.

Eckart Zitzler and Simon Künzli, "**Indicator-based Selection in Multiobjective Search**, in Xin Yao et al. (editors), *Parallel Problem Solving from Nature - PPSN VIII*, Springer-Verlag, Lecture Notes in Computer Science, Vol. 3242, pp. 832–842, Birmingham, UK, September 2004.

## SMS-EMOA

Emmerich et al. [2005] proposed an approach based on NSGA-II and the archiving techniques proposed by Knowles, Corne and Fleischer. This approach was called *S Metric Selection Evolutionary Multiobjective Algorithm*.

SMS-EMOA creates an initial population and generates a single solution per iteration (i.e., it uses steady state selection) using the crossover and mutation operators from NSGA-II. Then, it applies Pareto ranking. When the last nondominated front has more than one solution, SMS-EMOA uses hypervolume to decide which solution should be removed.

Michael Emmerich, Nicola Beume and Boris Naujoks, "**An EMO Algorithm Using the Hypervolume Measure as Selection Criterion**, in Carlos A. Coello Coello et al. (editors), *Evolutionary Multi-Criterion Optimization. Third International Conference, EMO 2005*, pp. 62–76, Springer. Lecture Notes in Computer Science Vol. 3410, Guanajuato, México, March 2005.

## SMS-EMOA

Beume et al. [2007] proposed a new version of SMS-EMOA in which the hypervolume contribution is not used when, in the Pareto ranking process, we obtain more than one front. In this case, they use the number of solutions that dominate to a certain individual (i.e., the solution that is dominated by the largest number of solutions is removed).

The authors of this approach indicate that their motivation to use the hypervolume is to improve the distribution of solutions along the Pareto front (in other words, hypervolume is used only as a density estimator).

Nicola Beume, Boris Naujoks and Michael Emmerich, "**SMS-EMOA: Multiobjective selection based on dominated hypervolume**", *European Journal of Operational Research*, Vol. 181, No. 3, pp. 1653–1669, 16 September, 2007.

## MO-CMA-ES

This is a multi-objective version of the **covariance matrix adaptation evolution strategy** (CMA-ES). It was proposed by Igel et al. [2007].

Its selection mechanism is based on a nondominated sorting that adopts as its second selection criterion either the crowding distance or the hypervolume contribution (two versions of the algorithm were tested, and the one based on the hypervolume has the best overall performance).

This MOEA is rotation invariant, as the original single-objective optimizer on which it is based.

Christian Igel, Nikolaus Hansen and Stefan Roth, "**Covariance Matrix Adaptation for Multi-objective Optimization**", *Evolutionary Computation*, Vol. 15, No. 1, pp. 1–28, Spring 2007.

# Recent Approaches

## SPAM

The *Set Preference Algorithm for Multiobjective optimization* is a generalization of IBEA which allows to adopt any set preference relation in its selection mechanism [Zitzler et al., 2008].

Eckart Zitzler, Lothar Thiele and Johannes Bader, "**SPAM: Set Preference Algorithm for Multiobjective Optimization**", in Günter Rudolph et al. (editors), *Parallel Problem Solving from Nature–PPSN X*, pp. 847–858, Springer, Lecture Notes in Computer Science Vol. 5199, Dortmund, Germany, September 2008.

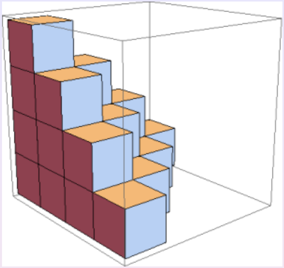
# Recent Approaches

## HyPE

The *hypervolume estimation algorithm for multi-objective optimization*, was proposed by Bader and Zitzler [2011]. In this case, the author proposes a quick search algorithm that uses Monte Carlo simulations to approximate the hypervolume contributions.

The core idea is that the actual hypervolume contribution value is not that important, but only the actual ranking that is produced with it. Although this proposal is quite interesting, in practice its performance is rather poor with respect that of MOEAs that use the exact hypervolume contributions.

Johannes Bader and Eckart Zitzler, "**HypE: An Algorithm for Fast Hypervolume-Based Many-Objective Optimization**", *Evolutionary Computation*, Vol. 19, No. 1, pp. 45–76, Spring, 2011.



## The Hypervolume

The **hypervolume** (also known as the *S* metric or the Lebesgue measure) of a set of solutions, measures the size of the portion of objective space that is dominated by such solutions, collectively.

The hypervolume is the only performance indicator that is known to be monotonic with respect to Pareto dominance. This guarantees that the true Pareto front achieves the maximum possible hypervolume value, and any other set will produce a lower value for this indicator.

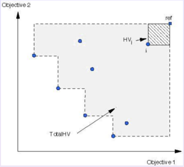
## The Hypervolume

Fleischer [2003] proved that, given a finite search space and a reference point, maximizing the hypervolume is equivalent to obtaining the Pareto optimal set. Therefore, a bounded set that contains the maximum possible hypervolume value for a certain population size, will only consist of Pareto optimal solutions.

This has been experimentally validated [Knowles, 2003; Emmerich, 2005], and it has been observed that such solutions also have a good distribution along the Pareto front.

M. Fleischer, "**The Measure of Pareto Optima. Applications to Multi-objective Metaheuristics**", in Carlos M. Fonseca et al. (editors), *Evolutionary Multi-Criterion Optimization. Second International Conference, EMO 2003*, pp. 519–533, Springer. Lecture Notes in Computer Science. Volume 2632, Faro, Portugal, April 2003.

# Recent Approaches

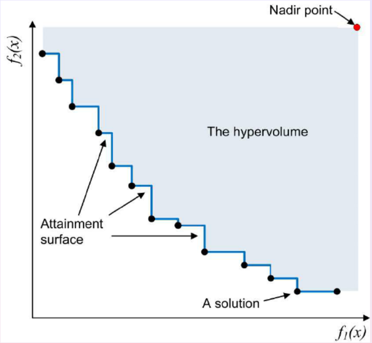



## The Hypervolume

The computation of the hypervolume depends on the reference point that we adopt, and this point can have a significant influence on the results. Some researchers have proposed to use the worst objective function values available in the current population, but this requires a scaling of the objectives.

However, the main drawback of using the hypervolume is its high computational cost. The best known algorithms currently available to compute the hypervolume have a complexity that is polynominal on the number of points, but such a complexity grows exponentially with the number of objectives.

# Recent Approaches




## The Hypervolume

The fact that no algorithm of polynomial complexity exists for computing the hypervolume in an exact manner, gave rise to the hypothesis that such an algorithm may not be at all possible. This is remarkable if we consider that the tight lower bound for the complexity of the hypervolume computation is $O(N \log N)$ [Beume, 2007].

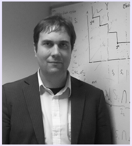
## The Hypervolume

Recent theoretical results strenghten this hypothesis: Bringmann and Friedrich [2009] proved that computing the hypervolume is #P-Complete. This means that no polynomial complexity algorithm exists, because otherwise, this would imply that $NP = P$.

Nevertheless, this indicator has triggered a significant amount of research. See for example:

- `http://ls11-www.cs.uni-dortmund.de/rudolph/hypervolume/start`
- `http://people.mpi-inf.mpg.de/~tfried/HYP/`
- `http://iridia.ulb.ac.be/~manuel/hypervolume`

# Recent Approaches



## The Hypervolume

It is worth noting that the use of the hypervolume to select solutions is not straightforward. This indicator operates on a set of solutions, and the selection operator considers only one solution at a time. Therefore, when using the hypervolume to select solutions, a fitness assignment strategy is required.

The strategy that has been most commonly adopted in the specialized literature consists of performing first a nondominated sorting procedure and then ranking the solutions within each front based on the hypervolume loss that results from removing a particular solution [Knowles and Corne, 2003; Emmerich et al., 2005; Igel et al., 2007; Bader et al., 2010].

### The Hypervolume

The main motivation for using indicators in the selection mechanism is scalability (in objective function space). However, the high computational cost of the hypervolume has motivated the exploration of alternative performance indicators, such as $\Delta_p$.

Oliver Schütze, Xavier Esquivel, Adriana Lara and Carlos A. Coello Coello, **Using the Averaged Hausdorff Distance as a Performance Measure in Evolutionary Multi-Objective Optimization**, *IEEE Transactions on Evolutionary Computation*, Vol. 16, No. 4, pp. 504–522, August 2012.

# Recent Approaches

## $\Delta_p$ Indicator

The $\Delta_p$ indicator can be seen as an "averaged Hausdorff distance" between our approximation and the true Pareto front. $\Delta_p$ combines some slight variations of two well-known performance indicators: generational distance [Van Veldhuizen, 1999] and inverted generational distance [Coello & Cruz, 2005].

$\Delta_p$ is a pseudo-metric that simultaneously evaluates proximity to the true Pareto front and the distribution of solutions along it. Although it is not a Pareto compliant indicator, in practice, it seems to work reasonably well, being able to deal with outliers. This makes it attractive as a performance indicator. Additionally, its computational cost is very low.

## $\Delta_p$ Indicator

Nevertheless, it is worth mentioning that in order to incorporate $\Delta_p$ in the selection mechanism of a MOEA, it is necessary to have an approximation of the true Pareto front at all times. This has motivated the development of techniques that can produce such an approximation in an efficient and effective manner.

For example, Gerst et al. [2011] lineralized the nondominated front produced by the current population and used that information in the so-called $\Delta_p$-EMOA, which was used to solve bi-objective problems. This algorithm is inspired on the SMS-EMOA and adopts an external archive.

K. Gerstl, G. Rudolph, O. Schütze and H. Trautmann, "**Finding Evenly Spaced Fronts for Multiobjective Control via Averaging Hausdorff-Measure**", in *The 2011 8th International Conference on Electrical Engineering, Computer Science and Automatic Control (CCE'2011)*, pp. 975–980, IEEE Press, Mérida, Yucatán, México, October 2011.

# Recent Approaches

## $\Delta_p$ Indicator

There was a further extension of this MOEA for dealing with problems having three objectives [Trautmann, 2012]. In this case, the algorithm requires some prior steps, which include reducing the dimensionality of the nondominated solutions and computing their convex hull.

This version of the $\Delta_p$-EMOA generates solutions with a better distribution, but requires more parameters and has a high computational cost when is used for solving many-objective optimization problems.

Heike Trautmann, Günter Rudolph, Christian Dominguez-Medina and Oliver Schütze, "**Finding Evenly Spaced Pareto Fronts for Three-Objective Optimization Problems**, in Oliver Schütze et al. (editors), *EVOLVE - A Bridge between Probability, Set Oriented Numerics, and Evolutionary Computation II*, pp. 89–105, Springer, Advances in Intelligent Systems and Computing Vol. 175, Berlin, Germany, 2012, ISBN 978-3-642-31519-0.

## $\Delta_p$ Indicator

Another possible way of incorporating $\Delta_p$ into a MOEA is to use the currently available nondominated solutions in a stepped way, in order to build an approximation of the true Pareto front.

This was the approach adopted by the $\Delta_p$-DDE [Rodríguez & Coello, 2012], which uses differential evolution as its search engine. This MOEA provides results of similar quality to those generated by SMS-EMOA, but at a much lower computational cost (in high dimensionality). Its main limitation is that its solutions are normally not well-distributed in many-objective problems. Additionally, it has difficulties to deal with disconnected Pareto fronts.


Cynthia A. Rodríguez Villalobos and Carlos A. Coello Coello, "**A New Multi-Objective Evolutionary Algorithm Based on a Performance Assessment Indicator**", in *2012 Genetic and Evolutionary Computation Conference (GECCO'2012)*, pp. 505–512, ACM Press, Philadelphia, USA, July 2012, ISBN: 978-1-4503-1177-9.

# Recent Approaches

## *R2*

Recently, some researchers have recommended the use of the *R*2 indicator, which was originally proposed by Hansen (1998) for comparing sets of solutions using utility functions [Brockhoff, 2012]. A utility function is a model of the decision maker preferences that maps each point from the objective function space to a utility value.

Dimo Brockhoff, Tobias Wagner and Heike Trautmann, "**On the Properties of the *R*2 Indicator**, in *2012 Genetic and Evolutionary Computation Conference (GECCO'2012)*, pp. 465–472, ACM Press, Philadelphia, USA, July 2012, ISBN: 978-1-4503-1177-9.

## $R2$

It is worth indicating that $R2$ is weakly monotonic and that it's correlated to the hypervolume, but has a much lower computational cost. Due to these properties, its use is recommended for dealing with many-objective problems. Nevertheless, the utility functions that are required to compute this indicator have to be properly scaled.

According to Brockhoff [2012], the unary version of the $R2$ indicator for a constant reference set can be expressed as follows:

$$R2(A, U) = -\frac{1}{|U|} \sum_{u \in U} \max_{\mathbf{a} \in A} \{u(\mathbf{a})\}, \tag{10}$$

where $A$ is the Pareto set approximation and $U$ is a set of utility functions.

With respect to the choice of the utility functions $u : \mathbb{R}^m \rightarrow \mathbb{R}$, there are several posibilities: weighted linear, weighted Tchebycheff or augmented Tchebycheff functions.

# Recent Approaches

## *R2*

Currently, there are already several MOEAs based on *R2*.

Raquel Hernández Gómez and Carlos A. Coello Coello, **MOMBI: A New Metaheuristic for Many-Objective Optimization Based on the R2 Indicator**, in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*, pp. 2488–2495, IEEE Press, Cancún, México, 20-23 June, 2013, ISBN 978-1-4799-0454-9.

Dimo Brockhoff, Tobias Wagner and Heike Trautmann, ***R2* Indicator-Based Multiobjective Search**, *Evolutionary Computation*, Vol. 23, No. 3, pp. 369–395, Fall 2015.

Alan Díaz-Manríquez, Gregorio Toscano-Pulido, Carlos A. Coello Coello and Ricardo Landa-Becerra, **A Ranking Method Based on the R2 Indicator for Many-Objective Optimization**, in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*, pp. 1523–1530, IEEE Press, Cancún, México, 20-23 June, 2013, ISBN 978-1-4799-0454-9.

# Recent Approaches

## *R2*

Dúng H. Phan and Junichi Suzuki, **R2-IBEA: R2 Indicator Based Evolutionary Algorithm for Multiobjective Optimization**, in *2013 IEEE Congress on Evolutionary Computation (CEC'2013)*, pp. 1836–1845, IEEE Press, Cancún, México, 20-23 June, 2013, ISBN 978-1-4799-0454-9.

Raquel Hernández Gómez and Carlos A. Coello Coello, "**Improved Metaheuristic Based on the *R*2 Indicator for Many-Objective Optimization**", in *2015 Genetic and Evolutionary Computation Conference (GECCO 2015)*, pp. 679–686, ACM Press, Madrid, Spain, July 11-15, 2015, ISBN 978-1-4503-3472-3.

# Recent Approaches

## NSGA-III

The *Nondominated Sorting Genetic Algorithm III* (NSGA-III) was proposed by Deb and Jain [2014] as an extension of NSGA-II specifically designed to deal with many-objective problems (i.e., multi-objective optimization problems having 4 or more objectives). NSGA-III still uses nondominated sorting (producing different levels), but in this case, the density estimation is done through adaptively updating a number of well-spread reference points.

Kalyanmoy Deb and Himanshu Jain, "**An Evolutionary Many-Objective Optimization Algorithm Using Reference-Point-Based Nondominated Sorting Approach, Part I: Solving Problems With Box Constraints**", *IEEE Transactions on Evolutionary Computation*, Vol. 18, No. 4, pp. 577–601, August 2014.