Multi-Objective Optimization Using Metaheuristics

Carlos A. Coello Coello

ccoello@cs.cinvestav.mx
CINVESTAV-IPN
Evolutionary Computation Group (EVOCINV)
Computer Science Department
Av. IPN No. 2508, Col. San Pedro Zacatenco
México, D.F. 07360, MEXICO

Eindhoven University of Technology
Eindhoven, The Netherlands July 2023
LECTURE 4
Some examples of test suites that have been proposed in the specialized literature to evaluate single-objective evolutionary algorithms are the following:

- The 5 test problems from De Jong [1975] for unconstrained optimization.
Test Problems

The test problems proposed by Whitley et al. [1996] and Goldberg [1989].


The deceptive problems from Goldberg and Mühlenbein.

The 8 test problemas from Digalakis & Margaritis [2000].

The multimodal test problems from Levy [1981], the test problems from Corana [1987], the test problems from Freudenstein-Roth and the test problems from Goldstein-Price [1981].

Ackley’s function and Wirestrass’ function [Bäck et al., 1997].
A set of test problems to evaluate the performance of a MOEA should include the following features (both in genotypic and in phenotypic space):

- Continuous vs. discontinuous vs. discrete
- Differentiable vs. non-differentiable
- Convex vs. concave
- Modality (unimodal, multi-modal)
- Numerical vs. alphanumeric
- Quadratic vs. nonquadratic
- Type of constraints (equalities, inequalities, linear, nonlinear)
- Low vs. high dimensionality (genotype, phenotype)
- Deceptive vs. nondeceptive
- Biased vs. unbiased portions of the true Pareto front
Test problems should range in difficulty from “easy” to “hard” as well as attempt to represent generic real-world situations.

Dynamically changing environments can include “moving cones” [Morrison & de Jong, 1999] with movement ranging from predictable to chaotic to non-stationary and deceptive.

One should also consider the following guidelines suggested by Whitley et al. [1996] in developing generic test suites:

- Some test suite problems are resistant to simple search strategies.
- Test suites contain nonlinear, unseparable & unsymmetric problems.
- Test suites contain scalable problems.
- Some test suite problems have scalable evaluation cost.
- Test problems have a canonical representation (ease of use).

Ideally, test problems used to evaluate a MOEA should contain features and difficulties similar to those found in the real-world problem(s) that we aim to solve.

However, the specialized literature presents a wide number of “artificial” test problems that emphasize certain aspects that are indeed difficult for most MOEAs, but that don’t necessarily represent the difficulties found in real-world problems.
MOP 1: This is the first test problem used by David Schaffer. Historically, it has a very high relevance, because it was the first test problem proposed to evaluate the performance of a MOEA. However, this problem is so simple that its Pareto front can be obtained in an analytic form. $PF_{true}$ is convex and the problem has a single decision variable.

$$F = (f_1(x), f_2(x))$$, where

$$f_1(x) = x^2$$,

$$f_2(x) = (x - 2)^2$$

where: $-10^5 \leq x \leq 10^5$
Unconstrained Problems

$P_{\text{true}}$ of MOP 1
Test Problems

Unconstrained Problems

$PF_{true}$ of MOP 1
MOP 2: This is the second test problem proposed by Fonseca. It is scalable. It is possible to add decision variables to this test problem without changing the shape of $PF_{true}$ (the Pareto front is concave in this case).

$$F = (f_1(\vec{x}), f_2(\vec{x}))$$

where:

$$f_1(\vec{x}) = 1 - \exp\left(-\sum_{i=1}^{n}(x_i - \frac{1}{\sqrt{n}})^2\right),$$

$$f_2(\vec{x}) = 1 - \exp\left(-\sum_{i+1}^{n}(x_i + \frac{1}{\sqrt{n}})^2\right)$$

where: $-4 \leq x_i \leq 4; \ i = 1, 2, 3$
Test Problems

Unconstrained Problems

$P_{\text{true}}$ of MOP 2
Test Problems

Unconstrained Problems

$PF_{\text{true}}$ of MOP 2
Test Problems

Unconstrained Problems

**MOP 3**: Proposed by Carlo Poloni. Both $P_{true}$ and $PF_{true}$ are disconnected.

Maximize $F = (f_1(x, y), f_2(x, y))$, where

$$
 f_1(x, y) = -[1 + (A_1 - B_1)^2 + (A_2 - B_2)^2], \\
 f_2(x, y) = -[(x + 3)^2 + (y + 1)^2]
$$

where: $-3.1416 \leq x, y \leq 3.1416$,

$$
 A_1 = 0.5 \sin 1 - 2 \cos 1 + \sin 2 - 1.5 \cos 2, \\
 A_2 = 1.5 \sin 1 - \cos 1 + 2 \sin 2 - 0.5 \cos 2, \\
 B_1 = 0.5 \sin x - 2 \cos x + \sin y - 1.5 \cos y, \\
 B_2 = 1.5 \sin x - \cos x + 2 \sin y - 0.5 \cos y
$$
Test Problems

Unconstrained Problems

$P_{true}$ of MOP 3
Unconstrained Problems

$PF_{true}$ of MOP 3
MOP 4: Proposed by Kursawe. There are disconnected and asymmetrical portions in $P_{true}$. $PF_{true}$ consists of 3 disconnected curves. It allows the use of an arbitrary number of decision variables, although scaling this test problem changes the shape of $PF_{true}$.

$$F = (f_1(\vec{x}), f_2(\vec{x}))$$

where:

$$f_1(\vec{x}) = \sum_{i=1}^{n-1} (-10e^{(-0.2)*\sqrt{x_i^2+x_{i+1}^2}}),$$

$$f_2(\vec{x}) = \sum_{i=1}^{n} (|x_i|^a + 5 \sin(x_i)^b)$$

where: $-5 \leq x_i \leq 5; \ i = 1, 2, 3; a = 0.8, b = 3$
Unconstrained Problems

$P_{true}$ of MOP 4
Test Problems

Unconstrained Problems

$PF_{true}$ of MOP 4
Test Problems

Unconstrained Problems

**MOP 5**: Proposed by Viennet. It has disconnected regions in $P_{true}$. $PF_{true}$ is a three-dimensional curve.

\[ F = (f_1(x, y), f_2(x, y), f_3(x, y)), \text{ where} \]

\[ f_1(x, y) = 0.5 \times (x^2 + y^2) + \sin(x^2 + y^2), \]

\[ f_2(x, y) = \frac{(3x - 2y + 4)^2}{8} + \frac{(x - y + 1)^2}{27} + 15, \]

\[ f_3(x, y) = \frac{1}{(x^2 + y^2 + 1)} - 1.1e^{(-x^2 - y^2)} \]

where: $-30 \leq x, y \leq 30$
Unconstrained Problems

$P_{true}$ of MOP 5
Unconstrained Problems

$PF_{true}$ of MOP 5
MOP 6: Proposed by Deb. Both $P_{true}$ and $PF_{true}$ are disconnected.

$$F = (f_1(x, y), f_2(x, y))$$

where

$$f_1(x, y) = x,$$

$$f_2(x, y) = (1 + 10y) \times$$

$$[1 - \left( \frac{x}{1 + 10y} \right)^\alpha - \frac{x}{1 + 10y} \sin(2\pi qx)]$$

where: $0 \leq x, y \leq 1$,

$$q = 4,$$

$$\alpha = 2$$
Unconstrained Problems

$P_{true}$ of MOP 6
Test Problems

Unconstrained Problems

$PF_{true}$ of MOP 6
**Unconstrained Problems**

**MOP 7**: Proposed by Viennet. $P_{true}$ is connected and $PF_{true}$ is a surface. This problem is relatively easy to solve by any MOEA.

\[ F = (f_1(x, y), f_2(x, y), f_3(x, y)) \], where

\[
\begin{align*}
    f_1(x, y) &= \frac{(x - 2)^2}{2} + \frac{(y + 1)^2}{13} + 3, \\
    f_2(x, y) &= \frac{(x + y - 3)^2}{36} + \frac{(-x + y + 2)^2}{8} - 17, \\
    f_3(x, y) &= \frac{(x + 2y - 1)^2}{175} + \frac{(2y - x)^2}{17} - 13
\end{align*}
\]

where: $-400 \leq x, y \leq 400$
Test Problems

Unconstrained Problems

$P_{true}$ of MOP 7
Test Problems

Unconstrained Problems

$PF_{true}$ of MOP 7
Historically, constraints have been handled in MOEAs through the use of penalty functions [Richardson et al., 1989]. However, many other methods to handle constraints are currently available, although few of them have been specifically designed for MOEAs.


Test Problems

Constrained Problems

**MOP-C1**: Proposed by Binh. In this case, $P_{true}$ is an area and $PF_{true}$ is a single convex curve.

\[ F = (f_1(x, y), f_2(x, y)), \text{ where} \]

\[
\begin{align*}
    f_1(x, y) &= 4x^2 + 4y^2, \\
    f_2(x, y) &= (x - 5)^2 + (y - 5)^2
\end{align*}
\]

where:

\[
\begin{align*}
    0 &\leq x \leq 5, \quad 0 \leq y \leq 3 \\
    0 &\geq (x - 5)^2 + y^2 - 25, \\
    0 &\geq -(x - 8)^2 - (y + 3)^2 + 7.7
\end{align*}
\]
Test Problems

Constrained Problems

$P_{true}$ of MOP-C1
Constrained Problems

$PF_{true}$ of MOP-C1
**Test Problems**

**Constrained Problems**

**MOP-C2:** Proposed by Osyczka. Both $P_{true}$ and $PF_{true}$ are disconnected.

\[
\begin{align*}
  f_1(\vec{x}) &= -(25(x_1 - 2)^2 + (x_2 - 2)^2 + (x_3 - 1)^2 \\
                   & \quad + (x_4 - 4)^2 + (x_5 - 1)^2, \\
  f_2(\vec{x}) &= x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2
\end{align*}
\]

\[0 \leq x_1, x_2, x_6 \leq 10, \ 1 \leq x_3, x_5 \leq 5, \ 0 \leq x_4 \leq 6,\]

\[0 \leq x_1 + x_2 - 2,\]

\[0 \leq 6 - x_1 - x_2,\]

\[0 \leq 2 - x_2 + x_1,\]

\[0 \leq 2 - x_1 + 3x_2,\]

\[0 \leq 4 - (x_3 - 3)^2 - x_4\]

\[0 \leq (x_5 - 3)^2 + x_6 - 4\]
Test Problems

Constrained Problems

$P_{true}$ of MOP-C2
Test Problems

Constrained Problems

$PF_{true}$ of MOP-C2
**Test Problems**

**Constrained Problems**

**MOP-C3**: Proposed by Viennet. $P_{true}$ is connected but it's asymmetrical. $PF_{true}$ is a 3D curve.

\[
\begin{align*}
  f_1(x, y) &= \frac{(x - 2)^2}{2} + \frac{(y + 1)^2}{13} + 3, \\
  f_2(x, y) &= \frac{(x + y - 3)^2}{175} + \frac{(2y - x)^2}{17} - 13, \\
  f_3(x, y) &= \frac{(3x - 2y + 4)^2}{8} + \frac{(x - y + 1)^2}{27} + 15
\end{align*}
\]

$-4 \leq x, y \leq 4$,

\[
\begin{align*}
  y &< -4x + 4, \\
  x &> -1, \\
  y &> x - 2
\end{align*}
\]
Test Problems

Constrained Problems

$P_{true}$ of MOP-C3
Constrained Problems

$PF_{true}$ of MOP-C3
Test Problems

Constrained Problems

**MOP-C4**: Proposed by Tanaka. $P_{true}$ is connected, but $PF_{true}$ is disconnected.

\[
\begin{align*}
  f_1(x, y) &= x, \\
  f_2(x, y) &= y
\end{align*}
\]

\[
0 < x, y \leq \pi,
\]

\[
0 \geq - (x^2) - (y^2) + 1 + (a \cos (b \arctan(x/y)))
\]

\[
a = 0.1 \\
b = 16
\]
Test Problems

Constrained Problems

$P_{true}$ of MOP-C4
Test Problems

Constrained Problems

$PF_{true}$ of MOP-C4
Test Problems Generators

MOP test functions can also be generated by using the single-objective functions. A methodology for constructing MOPs exhibiting desired characteristics has been proposed by Deb [1999].


He points out that when computationally derived a non-uniform distribution of vectors may exist in some Pareto front. He limits his initial test construction efforts to unconstrained MOPs of only two functions; his construction methodology then places restrictions on the two component functions so that resultant MOPs exhibit desired properties. To accomplish this he defines various generic bi-objective optimization problems, such as the example of the next slide.
Test Problems

Test Problems Generators

Minimize \( F = (f_1(\vec{x}), f_2(\vec{x})) \), where

\[
\begin{align*}
    f_1(\vec{x}) &= f(x_1, \ldots, x_m), \\
    f_2(\vec{x}) &= g(x_{m+1}, \ldots, x_N) h(f(x_1, \ldots, x_m), g(x_{m+1}, \ldots, x_N))
\end{align*}
\]

(1)

where function \( f_1 \) is a function of \((m < N)\) decision variables and \( f_2 \) a function of all \( N \) decision variables.

The function \( g \) is one of \((N - m)\) decision variables which are not included in function \( f \).

The function \( h \) is directly a function of \( f \) and \( g \) function values. The \( f \) and \( g \) functions are also restricted to positive values in the search space, i.e., \( f > 0 \) and \( g > 0 \).
Test Problems

Test Problems Generators

Deb lists five functions each for possible $f$ and $g$ instantiation, and four for $h$. These functions may then be “mixed and matched” to create MOPs with desired characteristics.

He states these functions have the following general effect:

- $f$ – This function controls vector representation uniformity along the Pareto front.
- $g$ – This function controls the resulting MOP’s characteristics – whether it is multifrontal or has an isolated optimum.
- $h$ – This function controls the resulting Pareto front’s characteristics (e.g., convex, disconnected, etc.)

These functions respectively influence search along and towards the Pareto front, and the shape of a Pareto front in $\mathbb{R}^2$. Deb implies that a MOEA has difficulty finding $PF_{true}$ because it gets “trapped” in a local Pareto front.
Test Problems

Test Problems Generators

**MOP-G1**: This is an example of the test problems generated with Deb’s methodology. In this case, $PF_{true}$ is convex.

\[
\begin{align*}
f_1(x_1) &= x_1, \\
f_2(\vec{x}) &= g(1 - \sqrt{(f_1/g)}) \\
g(\vec{x}) &= 1 + 9 \sum_{i=2}^{m} \frac{x_i}{(m-1)}
\end{align*}
\]

$m = 30; 0 \leq x_i \leq 1$
Test Problems

Test Problems Generators

For constrained test MOPs, Deb [2001] suggests extending his methodology in the following way:

\[
\begin{align*}
  f_1(\vec{x}) &= x_1 \\
f_2(\vec{x}) &= g(\vec{x}) \exp(-f_1(\vec{x})/g(\vec{x}))
\end{align*}
\]

subject to:

\[
c_j(x) = f_2(\vec{x}) - a_j \exp(-b_j f_1(\vec{x})) \geq 0, \quad j = 1, 2, \ldots, J
\]

There are $J$ inequalities, each of which has 2 parameters $(a_j, b_j)$, which makes that part of feasible region of the original (unconstrained) problem is now infeasible.
An example of this methodology is the following:

Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

\[
\begin{align*}
f_1(\vec{x}) & = x_1 \\
f_2(\vec{x}) & = (1 + x_2)/x_1
\end{align*}
\]

subject to:

\[
\begin{align*}
c_1(\vec{x}) & = x_2 + 9x_1 \geq 6 \\
c_2(\vec{x}) & = -x_2 + 9x_1 \geq 1
\end{align*}
\]
In fact, the next generic form is suggested: Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

\[
\begin{align*}
  f_1(\vec{x}) &= x_1 \\
  f_2(\vec{x}) &= g(\vec{x})(1 - f_1(\vec{x})/g(\vec{x})
\end{align*}
\]

subject to:

\[
\begin{align*}
  c_j(\vec{x}) &= \cos(\theta)(f_2(\vec{x}) - e) - \sin(\theta)f_1(\vec{x}) \geq \\
  &a|\sin(b\pi\sin(\theta))(f_2(\vec{x}) - e) + \cos(\theta)f_1(\vec{x}))|^d, \\
  j &= 1, 2, \ldots, J
\end{align*}
\]
Test Problems

Test Problems Generators

With 6 parameters \((\theta, a, c, d, e)\), \(x_1\) is restricted to the range \([0,1]\) and \(g(\vec{x})\) determines the bounds of the other decision variables.

Selecting values for the 6 parameters, we can generate different fitness landscapes.

It is worth noting that \(d\) controls the length of the continuous region of the Pareto front. As we decrease this region, a MOEA will tend to find less points of \(PF_{true}\) because of the discretization of \(\vec{x}\).
If we increase the value of $a$, the length of the “cuts” becomes more profound, which requires the search to proceed through a narrowed corridor. Evidently, this makes more difficult the search.

We can also depart from the periodic disconnected regions of $PF_{true}$ by changing $c$ from its initial value of 1.

$\theta$ and $e$ control the slope and the change of direction of $PF_{true}$, respectively.
Test Problems

Zitzler-Deb-Thiele (ZDT) Test Problems

Each of the test problems shown next is structured in the same way and it consists of 3 functions $f_1$, $g$, $h$:

\[
\text{Minimize : } F(\vec{x}) = (f_1, f_2), \\
\text{subject to : } f_2(\vec{x}) = g(x_2, \ldots, x_m)h(f_1(x_1), g(x_2, \ldots, x_m)), \\
\text{where : } \vec{x} = (x_1, \ldots, x_M). \tag{5}
\]

$f_1$ is a function of only the first decision variable, $g$ is a function of the $m - 1$ remaining decision variables, and the parameters of $h$ are the values of $f_1$ and $g$. 
Zitzler-Deb-Thiele (ZDT) Test Problems


The test problems differ in these 3 functions and in the number of decision variables $m$, as well as in the values that the decision variables can take. These problems have been heavily used to validated MOEAs in the specialized literature.
ZDT Test Problems

$PF_{true}$ of ZDT1
ZDT Test Problems

$PF_{true}$ of ZDT2
ZDT Test Problems

$PF_{true}$ of ZDT3
ZDT Test Problems

$PF_{true}$ of ZDT4
ZDT Test Problems

$PF_{true}$ of ZDT5
ZDT Test Problems

$PF_{\text{true}}$ of ZDT6
Test Problems

Deb-Thiele-Laumanns-Zitzler (DTLZ) Test Problems

Another desirable feature of a test problem is that it can scale up to any number of dimensions.

Since the mapping between the genotypic and the phenotypic space can be considerably nonlinear, we can exploit this property to generate test problems with a high degree of difficulty.

Deb et al. [2002,2005] proposed the so-called Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite in which the problems are scalable to a number of objectives defined by the user. This test suite has also been very popular in the specialized literature.

DTLZ1

$\text{PF}_{\text{true}}$ is linear, separable and multimodal.

Minimize:

\[
f_1(x) = \frac{1}{2} x_1 x_2 \ldots x_{M-1} (1 + g(x_M)), \quad (6)
\]

\[
f_2(x) = \frac{1}{2} x_1 x_2 \ldots (1 - x_{M-1})(1 + g(x_M)), \quad (7)
\]

\[
\vdots \quad \vdots \quad (8)
\]

\[
f_{M-1}(x) = \frac{1}{2} x_1 (1 - x_2)(1 + g(x_M)), \quad (9)
\]

\[
f_M(x) = \frac{1}{2} (1 - x_1)(1 + g(x_M)), \quad (10)
\]

subject to

\[
0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, \ldots, n \quad (11)
\]

where:

\[
g(x_M) = 100 \left[ |x_M| + \sum_{x_i \in x_M} (x_i - 0.5)^2 - \cos(20 \pi (x_i - 0.5)) \right] \quad (12)
\]
DTLZ1

It is normally adopted with $M = 3$. The Pareto optimal set is located at $x^*_M = 0$ and the values of the objective functions at the linear hyperplane $\sum_{m=1}^{M} = 0.5$.

The search space contains $(11^k - 1)$ local Pareto fronts ($k$ is a value defined by the user, such that the number of decision variables is: $n = M + k - 1$. It is common to adopt $k = 5$).
Test Problems

DTLZ Test Problems

$PF_{true}$ of DTLZ1
DTLZ2

Minimize:

\begin{align*}
    f_1(x) &= (1 + g(x_M)) \cos(x_1 \pi / 2) \cos(x_2 \pi / 2) \ldots \cos(x_{M-2} \pi / 2) \cos(x_{M-1} \pi / 2), \\
    f_2(x) &= (1 + g(x_M)) \cos(x_1 \pi / 2) \cos(x_2 \pi / 2) \ldots \cos(x_{M-2} \pi / 2) \sin(x_{M-1} \pi / 2), \\
    f_3(x) &= (1 + g(x_M)) \cos(x_1 \pi / 2) \cos(x_2 \pi / 2) \ldots \sin(x_{M-2} \pi / 2), \\
    & \quad \vdots \\
    f_{M-1}(x) &= (1 + g(x_M)) \cos(x_1 \pi / 2) \sin(x_2 \pi / 2), \\
    f_M(x) &= (1 + g(x_M)) \sin(x_1 \pi / 2). \\
\end{align*}

subject to: \( 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, \ldots, n \)

where: \( g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2 \)
Test Problems

DTLZ2

The Pareto optimal set is located at: \( x_i = 0.5 \) for every \( x_i \in x_M \) and all the objective functions have to satisfy: \( \sum_{i=1}^{M} (f_i)^2 = 1 \). It is suggested to use \( k = |x_M| = 10 \).

The total number of decision variables is: \( n = M + k - 1 \).
DTLZ Test Problems

$PF_{true}$ of DTLZ2
DTLZ3

Minimize:

\[ f_1(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \cos(x_2 \pi/2) \ldots \cos(x_{M-2} \pi/2) \cos(x_{M-1} \pi/2), \]
\[ f_2(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \cos(x_2 \pi/2) \ldots \cos(x_{M-2} \pi/2) \sin(x_{M-1} \pi/2), \]
\[ f_3(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \cos(x_2 \pi/2) \ldots \sin(x_{M-2} \pi/2), \]
\[ \vdots \]
\[ f_{M-1}(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \sin(x_2 \pi/2), \]
\[ f_M(x) = (1 + g(x_M)) \sin(x_1 \pi/2). \]

subject to: \( 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, \ldots, n \)

where: \( g(x_M) = 100[|x_M| + \sum_{x_i \in x_M} (x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))] \)
DTLZ3

It is suggested that \( k = |x_M| = 10 \). There is a total of \( n = M + k - 1 \) decision variables.

The function \( g \) described before, introduces \((3k - 1)\) false Pareto fronts. All of these false Pareto fronts are parallel to the global Pareto front and, therefore, a MOEA can get easily trapped in one of them before converging to the Pareto optimal front which is located at \( g^* = 0 \).

The true Pareto front corresponds to \( x_M = (0.5, \ldots, 0.5)^T \).
DTLZ Test Problems

$PF_{true}$ of DTLZ3
DTLZ4

Minimize:

\[ f_1(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \cos(x_2 \pi/2) \cdots \cos(x_{M-2} \pi/2) \cos(x_{M-1} \pi/2) , \]
\[ f_2(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \cos(x_2 \pi/2) \cdots \cos(x_{M-2} \pi/2) \sin(x_{M-1} \pi/2) , \]
\[ f_3(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \cos(x_2 \pi/2) \cdots \sin(x_{M-2} \pi/2) , \]
\[ \vdots \quad \vdots \]
\[ f_{M-1}(x) = (1 + g(x_M)) \cos(x_1 \pi/2) \sin(x_2 \pi/2) , \]
\[ f_M(x) = (1 + g(x_M)) \sin(x_1 \pi/2). \]

subject to: \( 0 \leq x_i \leq 1 \ \forall \ i = 1, 2, \ldots, n \)

where: \( g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2 \)
DTLZ4

It is suggested to use $\alpha = 100$ in this case. Again, all the decision variables $x_1$ to $x_{M-1}$ are varied in the range $(0 : 1)$.

It is also suggested to use $k = 10$. There are $n = M + k - 1$ decision variables in this problem.

In this case, there is dense set of solutions close to the plane $f_M - f_1$. 
DTLZ Test Problems

$PF_{\text{true}}$ of DTLZ4
DTLZ5

Minimize:

\[ f_1(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2) \ldots \cos(\theta_{M-2} \pi/2) \cos(\theta_{M-1} \pi/2), \]
\[ f_2(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2) \ldots \cos(\theta_{M-2} \pi/2) \sin(\theta_{M-1} \pi/2), \]
\[ f_3(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2) \ldots \sin(\theta_{M-2} \pi/2), \]
\[ \vdots \]
\[ f_{M-1}(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \sin(\theta_2 \pi/2), \]
\[ f_M(x) = (1 + g(x_M)) \sin(\theta_1 \pi/2). \]

subject to: \( 0 \leq x_i \leq 1 \) \( \forall \) \( i = 1, 2, \ldots, n \)

where: \( \theta_i = \frac{\pi}{4(1 + g(x_M))} (1 + 2g(x_M)x_i), \) for \( i = 2, 3, \ldots, (M - 1) \)

\[ g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2 \]
Test Problems

DTLZ5

It is suggested to use function $g$ with $k = |x_M| = 10$. Also, there are $n = M + k - 1$ decision variables and the Pareto optimal set corresponds to $x_i = 0.5$ for every $x_i \in x_M$ and every objective function must satisfy:

$$\sum_{i=1}^{M} (f_i)^2 = 1.$$ 

This problem evaluates the capability of a MOEA to converge to a curve.

It is suggested to use ($M \in [5, 10]$).
DTLZ Test Problems

$PF_{true}$ of DTLZ5
DTLZ6

Minimize:

\[ f_1(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2) \ldots \cos(\theta_{M-2} \pi/2) \cos(\theta_{M-1} \pi/2), \]
\[ f_2(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2) \ldots \cos(\theta_{M-2} \pi/2) \sin(\theta_{M-1} \pi/2), \]
\[ f_3(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \cos(\theta_2 \pi/2) \ldots \sin(\theta_{M-2} \pi/2), \]
\[ \vdots \]
\[ f_{M-1}(x) = (1 + g(x_M)) \cos(\theta_1 \pi/2) \sin(\theta_2 \pi/2), \]
\[ f_M(x) = (1 + g(x_M)) \sin(\theta_1 \pi/2). \]

subject to: \(0 \leq x_i \leq 1\) \(\forall i = 1, 2, \ldots, n\)

where: \(\theta_i = \frac{\pi}{4(1 + g(x_M))}(1 + 2g(x_M)x_i), \forall i = 2, 3, \ldots, (M - 1)\)

\[ g(x_M) = \sum_{x_i \in X_M} (x_i)^{0.1} \]
Test Problems

DTLZ6

The Pareto optimal set is located at $x_i = 0$ for every $x_i \in x_M$.

The size of the vector $x_M$ is chosen as 10 and the total number of decision variables is identical to the one used for DTLZ5.
DTLZ Test Problems

$PF_{true}$ of DTLZ6
DTLZ7

Minimize:

\[ f_1(x) = x_1, \]
\[ f_2(x) = x_2, \]
\[ \vdots \]
\[ f_{M-1}(x) = x_{M-1} \]
\[ f_M(x) = (1 + g(x_M)) \cdot h(f_1, f_2, \ldots, f_{M-1}, g(x)) \]

subject to: \( 0 \leq x_i \leq 1 \) \( \forall \ i = 1, 2, \ldots, n \)

where: \( g(x) = 1 + \frac{9}{|x_M|} \sum_{x_i \in x_M} x_i, \)

\[ h(f_1, f_2, \ldots, f_{M-1}, g) = M - \sum_{i=1}^{M-1} \left( \frac{f_i}{1 + g(x)(1 + \sin(3\pi f_i))} \right) \]
DTLZ7

This problem has $2M - 1$ disconnected Pareto optimal regions.

$g$ requires $k = |x_{Mj}|$ decision variables and the total number of decision variables is $n = M + k - 1$. It is suggested to use $k = 20$.

The Pareto optimal set corresponds to $x_M = 0$.

This problem aims to test the ability of a MOEA to maintain, simultaneously, solutions at different regions of the search space.
DTLZ Test Problems

$PF_{true}$ of DTLZ7
Test Problems

DTLZ8

Minimize:

\[ f_j(x) = \frac{1}{\left\lfloor n/M \right\rfloor} \sum_{i=\left\{ \frac{(i-1)n}{M} \right\}}^{\left\lfloor \frac{n}{M} \right\rfloor} (x_i), \forall j = 1, 2, \ldots, M, \]

subject to: \( 0 \leq x_i \leq 1 \ \forall \ i = 1, 2, \ldots, n \)

where: \( g_j(x) = f_M(x) + 4f_j(x) - 1 \geq 0, \forall j = 1, 2, \ldots, (M - 1) \)

\( g_M(x) = 2f_M(x) + \min_{i,j=1,i\neq j}^{M-1} [f_i(x) + f_j(x)] - 1 \geq 0, \)
Test Problems

**DTLZ8**

The number of decision variables must be larger than the number of objectives $n > M$. It is suggested to use $n = 10M$.

This problem has $M$ constraints. The true Pareto front is a combination of a straight line and a hyperplane.

The straight line is the intersection of the first $(M - 1)$ constraints (with $f_1 = f_2 = \ldots = f_{M-1}$ and the hyperplane is represented through constraint $g_M$).
Test Problems

DTLZ Test Problems

$PF_{true}$ of DTLZ8
DTLZ9

Minimize:

\[ f_j(x) = \frac{1}{\lfloor n/M \rfloor} \sum_{i=(j-1)\lfloor n/M \rfloor}^{\lfloor j \cdot n/M \rfloor} (x_i^{0.1}) , \forall j = 1, 2, \ldots, M, \]

subject to: \( 0 \leq x_i \leq 1 \ \forall \ i = 1, 2, \ldots, n \)

where: \( g_j(x) = f_M^2(x) + f_j^2(x) - 1 \geq 0 , \forall j = 1, 2, \ldots, (M - 1) \)
Test Problems

DTLZ9

The number of decision variables must be larger than the number of objectives. It is suggested to use: $n = 10M$.

The true Pareto front is a curve with $f_1 = f_2 = \ldots = f_{M-1}$, similar to the Pareto front of DTLZ5. However, in this case, the density of solutions decreases as we approach the Pareto optimal region.

The Pareto front is at the intersection of all the $(M - 1)$ constraints, which can cause difficulties to a MOEA.
DTLZ Test Problems

$PF_{\text{true}}$ of DTLZ9
Okabe’s Test Problems

Tatsuya Okabe et al. [2004] proposed a methodology to generate multi-objective test problems based on a mapping of probability density functions from decision variable space to objective function space. They also provide two examples of this methodology.

The basic idea is to depart from an initial space (called $S^2$) between decision variable space and objective function space and from there, they build both spaces by applying appropriate functions to $S^2$. For this sake, the authors proposed to use the inverse of the generation operation (i.e., deformation, rotation and translation).

Okabe's Test Problems

OKA1:
Minimize:

\[ f_1 = x'_1, \]
\[ f_2 = \sqrt{2\pi} - \sqrt{|x'_1|} + 2|x'_2| - 3\cos(x'_1) - 3\frac{1}{2}, \]

where:

\[ x'_1 = \cos(\pi/12)x_1 - \sin(\pi/12)x_2, \]
\[ x'_2 = \sin(\pi/12)x_1 + \cos(\pi/12)x_2, \]

subject to:

\[ x_1 \in [6\sin(\pi/12), 6\sin(\pi/12) + 2\pi\cos(\pi/12)], \]
\[ x_2 \in [-2\pi\sin(\pi/12), 6\cos(\pi/12)], \]

(13)
Okabe’s Test Problems

The Pareto optimal set is located at: \( x'_2 = 3 \cos(x'_1 + 3) \) and \( x'_1 \in [0, 2\pi] \).

The Pareto front is located at: \( f_2 = \sqrt{2\pi} - \sqrt{f_1} \) and \( f_1 \in [-\pi, \pi] \).

The Distribution indicator is:

\[
D_{x \rightarrow f} = \frac{3}{2} \left| x'_2 - 3 \cos(x'_1) - 3 \right|^\frac{2}{3}
\]  

(14)

The Distribution indicator measures the amount of distortion that the probability density suffers in decision variable space under the mapping from decision variable space to objective function space.
Okabe’s Test Problems

Pareto front of OKA1
Test Problems

Okabe’s Test Problems

Pareto optimal set of OKA1

Carlos A. Coello Coello

Multi-Objective Optimization
Okabe’s Test Problems

**OKA2:**
Minimize:

\[
\begin{align*}
  f_1 &= x_1, \\
  f_2 &= 1 - \frac{1}{4\pi^2}(x_1 + \pi)^2 + |x_2 - 5 \cos(x_1)|^{\frac{1}{3}} + |\frac{x}{3} - 5 \sin(x_1)|^{\frac{1}{3}},
\end{align*}
\]

subject to:

\[
\begin{align*}
  x_1 &\in [-\pi, \pi], \\
  x_2, x_3 &\in [-5, 5]
\end{align*}
\]
Okabe’s Test Problems

The Pareto optimal set is located at: \((x_1, x_2, x_3) = (x_1, 5 \cos(x_1), 5 \sin(x_1))\) and \(x_1 \in [-\pi, \pi]\).

The true Pareto front is located at: \(f_2 = 1 - \frac{1}{4\pi^2} (f_1 + \pi)^2\) and \(f_1 \in [-\pi, \pi]\).

The Distribution indicator is: \(D_{x \rightarrow f} = 9|x_2 - 5 \cos(x_1)|^{\frac{2}{3}}|x_3 - 5 \sin(x_1)|^{\frac{2}{3}}\).
Okabe’s Test Problems

Pareto front of OKA2
Okabe’s Test Problems

Pareto optimal set of OKA2
Huband et al. [2006] proposed a methodology to design test problems which are quite challenging for MOEAs. The set that they used to exemplify their methodology is known as the Walking-Fish-Group (WFG) test suite.

In the next slides, we show the shapes for the objective functions and the transformation functions.

WFG Test Problems (Shapes for the objective functions)

\[
\text{linear}_1(x_1, \ldots, x_{M-1}) = \prod_{i=1}^{M-1} x_i
\]

\[
\text{linear}_{m=2:M-1}(x_1, \ldots, x_{M-1}) = \left( \prod_{i=1}^{M-m} x_i \right) (1 - x_{M-m+1})
\]

\[
\text{linear}_M(x_1, \ldots, x_{M-1}) = 1 - x_1
\]

\[
\text{convex}_1(x_1, \ldots, x_{M-1}) = \prod_{i=1}^{M-1} (1 - \cos(x_i \pi/2))
\]

\[
\text{convex}_{m=2:M-1}(x_1, \ldots, x_{M-1}) = \left( \prod_{i=1}^{M-m} (1 - \cos(x_i \pi/2)) \right) (1 - \sin(x_{M-m+1} \pi/2))
\]

\[
\text{convex}_M(x_1, \ldots, x_{M-1}) = 1 - \sin(x_1 \pi/2)
\]
WFG Test Problems (Shapes for the objective functions)

\[
\text{concave}_1(x_1, \ldots, x_{M-1}) = \prod_{i=1}^{M-1} \sin(x_i \pi / 2)
\]

\[
\text{concave}_{m=2:M-1}(x_1, \ldots, x_{M-1}) = \left( \prod_{i=1}^{M-m} \sin(x_i \pi / 2) \right) \cos(x_{M-m+1} \pi / 2)
\]

\[
\text{concave}_M(x_1, \ldots, x_{M-1}) = \cos(x_1 \pi / 2)
\]

\[
\text{mixed}_M(x_1, \ldots, x_{M-1}) = \left( 1 - x_1 - \frac{\cos(2Ax_1 + \pi / 2)}{2A\pi} \right)^{\alpha}
\]

\[
\text{disc}_M(x_1, \ldots, x_{M-1}) = 1 - x_1^{\alpha} \cos^2(Ax_1^{\beta} \pi)
\]
b\_poly(y, \alpha) = y^\alpha \\
b\_flat(y, A, B, C) = A + \min(0, \lfloor y - B \rfloor) \frac{A(B - y)}{B} - \min(0, \lfloor C - y \rfloor) \frac{(1 - A)(y - C)}{1 - C} \\
b\_param(y, u(\vec{y}'), A, B, C) = y^{B+(C-B)}(A-(1-2u(\vec{y}'))\lfloor 0.5-u(\vec{y}') \rfloor + A) \\
s\_linear(y, A) = \frac{|y - A|}{||A - y|| + A} \\
s\_decept(y, A, B, C) = 1 + (|y - A| - B) \left( \frac{|y - A + B| \left( 1 - C + \frac{A-B}{B} \right)}{A-B} + \frac{|A + B - y| \left( 1 - C + \frac{1-A-B}{B} \right)}{1 - A - B} + \frac{1}{B} \right) \\
s\_multi(y, A, B, C) = \frac{1 + \cos \left( (4A + 2)\pi \left( 0.5 - \frac{|y-C|}{2(\lfloor C-y \rfloor + C)} \right) \right) + 4B \left( \frac{|y-C|}{2(\lfloor C-y \rfloor + C)} \right)^2}{b + 2} \\
r\_sum(\vec{y}, \bar{w}) = \frac{\sum_{i=1}^{n} |\vec{y}| w_i y_i}{\sum_{i=1}^{n} |\vec{y}| w_i} \\
r\_nonsep(\vec{y}, A) = \frac{\sum_{j=1}^{n} \left( y_j + \sum_{k=0}^{A-2} |y_j - y_{1+(j+k)\mod n}| \right)}{|\vec{y}| A} \left( \frac{A}{n} \right)^2 \left( 1 + 2A - 2 \left\lfloor \frac{A}{2} \right\rfloor \right)
Test Problems

**WFG1:**
Minimize

\[
\begin{align*}
    f_{m=1:M-1}(\vec{x}) &= x_M + S_m\text{convex}_m(x_1, \ldots, x_{M-1}) \\
    f_M(\vec{x}) &= x_M + S_M\text{mixed}_M(x_1, \ldots, x_{M-1})
\end{align*}
\]

where

\[
\begin{align*}
    y_{i=1:M-1} &= \text{r_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [2((i-1)k/(M-1)+1), \ldots, 2ik/(M-1)]) \\
    y_M &= \text{r_sum}([y'_{k+1}, \ldots, y'_n], [2(k+1), \ldots, 2n]) \\
    y'_{i=1:n} &= \text{b_poly}(y''_{i}, 0.02) \\
    y''_{i=1:k} &= y''' \\
    y''_{i=k+1:n} &= \text{b_flat}(y''', 0.8, 0.75, 0.85) \\
    y'''_{i=1:k} &= z_i, [0,1] \\
    y'''_{i=k+1:n} &= \text{s_linear}(z_i, [0,1], 0.35)
\end{align*}
\]
Test Problems

For all problems:
The decision vector is $z = [z_1, \ldots, z_k, z_{k+1}, \ldots, z_n]$ where

\[ 0 \leq z_i \leq z_{i,\text{max}}. \]

\[
\begin{align*}
Z_{i=1:n,\text{max}} &= 2i \\
Z_{i=1:n,[0,1]} &= \frac{z_i}{z_{i,\text{max}}} \\
x_{i=1:M-1} &= \max(y_M, A_i)(y_i - 0.5) + 0.5 \\
x_M &= y_M \\
S_{m=1:M} &= 2m \\
A_1 &= 1 \\
A_{2:M-1} &= \begin{cases} 
0, & \text{for WFG3} \\
1, & \text{otherwise} 
\end{cases}
\]
Test Problems

WFG Test Problems

Pareto front of WFG1
WFG2:
Minimize
\[
\begin{align*}
    f_{m=1:M-1}(\bar{x}) &= x_M + S_{m\text{convex}}(x_1, \ldots, x_{M-1}) \\
    f_M(\bar{x}) &= x_M + S_{M\text{disc}}(x_1, \ldots, x_{M-1})
\end{align*}
\]
where
\[
\begin{align*}
    y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k}/(M-1)+1, \ldots, y'_{ik}/(M-1)], [1, \ldots, 1]) \\
    y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\
    y'_{i=1:k} &= y''_i \\
    y'_{i=k+1:k+l/2} &= \text{r\_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\
    y'_{i=1:k} &= z_i, [0,1] \\
    y''_{i=k+1:n} &= \text{s\_linear}(z_i, [0,1], 0.35)
\end{align*}
\]
WFG Test Problems

Pareto front of WFG2
WFG3:
Minimize

\[ f_{m=1:M}(\vec{x}) = x_M + S_{m\text{inear}}(x_1, \ldots, x_{M-1}) \]

where

\[
\begin{align*}
 y_{i=1:M-1} &= r_{-\text{sum}}([y'_{i-1}k/(M-1)+1, \ldots, y'_{i}k/(M-1)], [1, \ldots, 1]) \\
 y_M &= r_{-\text{sum}}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\
 y'_{i=1:k} &= y''_i \\
 y'_{i=k+1:k+l/2} &= r_{-\text{nonsep}}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\
 y''_i &= z_i, [0,1] \\
 y''_{i=k+1:n} &= s_{-\text{linear}}(z_i, [0,1], 0.35)
\end{align*}
\]
WFG Test Problems

Pareto front of WFG3
Test Problems

WFG4:
Minimize

\[ f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1}) \]

where

\[ y_{i=1:M-1} = \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \]
\[ y_M = \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \]
\[ y'_{i=1:n} = \text{s\_multi}(z_i,[0,1], 30, 10, 0.35) \]
Test Problems

WFG Test Problems

Pareto front of WFG4
WFG5:
Minimize

\[ f_{m=1:M}(\bar{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1}) \]

where

\[ y_{i=1:M-1} = \text{r_sum}( [y'_{(i-1)(k/(M-1)+1)}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \]
\[ y_M = \text{r_sum}( [y'_{k+1}, \ldots, y'_{n}], [1, \ldots, 1]) \]
\[ y'_{i=1:n} = \text{s_decept}(z_i, [0,1], 0.35, 0.001, 0.05) \]
WFG Test Problems

Pareto front of WFG5
WFG6:
Minimize
\[ f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1}) \]
where
\[
\begin{align*}
y_{i=1:M-1} & = r\_\text{nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\
y_M & = r\_\text{nonsep}([y'_{k+1}, \ldots, y'_n], l) \\
y'_{i=1:k} & = z_i, [0,1] \\
y'_{i=k+1:n} & = s\_\text{linear}(z_i, [0,1], 0.35)
\end{align*}
\]
WFG Test Problems

Pareto front of WFG6
**Test Problems**

WFG7:
Minimize

\[ f_{m=1:M}(\vec{x}) = x_M + S_m\text{concave}_m(x_1, \ldots, x_{M-1}) \]

where

\[
\begin{align*}
y_{i=1:M-1} & = \text{r}_\text{sum}([y'_{(i-1)k}/(M-1)+1, \ldots, y'_{ik}/(M-1)], [1, \ldots, 1]) \\
y_M & = \text{r}_\text{sum}([y'_{k+1}, \ldots, y'_{n}], [1, \ldots, 1]) \\
y'_{i=1:k} & = y''_i \\
y'_{i=k+1:n} & = \text{s}_\text{linear}(y''_i, 0.35) \\
y''_{i=1:k} & = \text{b}_\text{param}(Z_i,[0,1], \text{r}_\text{sum}([Z_{i+1}[0,1], \ldots, Z_n[0,1]], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_{i=k+1:n} & = Z_i,[0,1]
\end{align*}
\]
Test Problems

WFG Test Problems

Pareto front of WFG7
WFG8:
Minimize

\[ f_{m=1:M}(\bar{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1}) \]

where

\[
\begin{align*}
y_{i=1:M-1} &= \text{r_sum}([y_{i-1}'k/(M-1)+1, \ldots, y_{ik}'/(M-1)], [1, \ldots, 1]) \\
y_M &= \text{r_sum}([y_{k+1}', \ldots, y_n'], [1, \ldots, 1]) \\
y_{i=1:k}' &= y_i'' \\
y_{i=k+1:n}' &= \text{s_linear}(y_i'', 0.35) \\
y_{i=1:k}'' &= z_i, [0,1] \\
y_{i=k+1:n}'' &= \text{b_param}(z_i, [0,1], \text{r_sum}([z_1, [0,1], \ldots, z_{i-1}, [0,1]], [1, \ldots, 1]), 0.98/49.98, 0.02, 50)
\end{align*}
\]
WFG Test Problems

Pareto front of WFG8
WFG9:
Minimize

\[ f_{m=1:M}(\vec{x}) = x_M + S_{m\text{concave}}(x_1, \ldots, x_{M-1}) \]

where

\[
\begin{align*}
y_{i=1:M-1} &= \text{r\_nonsep}([y_{i-1}k/(M-1)+1, \ldots, y_{i-1}k/(M-1)], k/(M-1)) \\
y_M &= \text{r\_nonsep}([y_{k+1}, \ldots, y_n], l) \\
y'_{i=1:k} &= \text{s\_decept}(y''_i, 0.35, 0.001, 0.05) \\
y'_{i=k+1:n} &= \text{s\_multi}(y''_i, 30, 95, 0.35) \\
y''_{i=1:n-1} &= \text{b\_param}(z_i, [0,1], \text{r\_sum}([z_{i+1}, [0,1], \ldots, z_n, [0,1]], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_n &= z_n, [0,1]
\end{align*}
\]
WFG Test Problems

Pareto front of WFG9
Use of Lamé Superspheres

In order to design multi-objective test problems with different Pareto optimal fronts, Emmerich and Deutz [2007] introduced a scalable test suite based on the Lamé superspheres (LSS). Although this methodology is limited to design Pareto optimal geometries with spherical shapes, it can be considered as the first study focused on the Pareto shape of multi-objective test problems. Something remarkable about this proposal are the mirror test problems which adopt an inverted sphere as the Pareto shape of the proposed multi-objective test problems. Even though the use of mirror spheres had already been adopted as a Pareto optimal surface by Huband et al. [2006], the parameter $\gamma$ of the Lamé spheres is able to modify the convexity/concavity degree in these test problems.

Use of Lamé Superspheres

In addition to new Pareto optimal shapes, the Lamé superspheres test suite incorporates features such as multi-modality and many-to-one mapping which introduce additional difficulties for solving these problems using a MOEA. Since this test suite adopts the DTLZ framework, distance and position parameters can also be easily identified.


Complicated Pareto Sets

The Pareto optimal fronts for all these test problems are defined in one and two dimensions, i.e., they become degenerate for more than two and three objectives, respectively.

Regarding the Pareto optimal fronts, four continuous and connected surfaces including convexity and concavity generalize the Pareto shapes in this test suite. The convergence difficulties in this benchmark are specifically stated by the topology of the PSs.

The absence of multi-modality and non-separability, are the shortcomings in this test suite. However, the use of the modular approach in this testbed, makes difficult to determine position and distance parameters, which becomes an advantage over the previous test suites.
Large Scale MOPs

As pointed out by Huband et al. [2006], variable linkages should be considered in the construction of multi-objective test problems. This feature in test instances is particularly important because, it makes it more difficult for a MOEA to properly exploit optimal solutions.

Cheng et al. [2017] introduced a set of nine test problems specially designed to test MOEAs for large scale optimization (i.e., for multi-objective problems with a large number of decision variables).

Large Scale MOPs

In the **Large Scale Multi-Objective Problems** (LSMOPs), the variable dependencies are stated by two linear variable linkage functions (linear and nonlinear).

In addition to the dependencies among variables, this test suite introduces correlation between decision variables and objectives by means of a correlation matrix. Although the test problems are scalable to an arbitrary number of objectives, this test suite is limited to three Pareto optimal shapes, concretely, the PFs from DTLZ1 (normalized in objective function space), DTLZ2, and DTLZ7.
A Toolkit

Masuda et al. [2016] proposed a toolkit to generate scalable test problems. This test suite is mainly focused on the design of different Pareto optimal shapes. The methodology introduced in this approach allows the design of Pareto optimal surfaces by using a finite number of vertices. Such vertices state the Pareto optimal front whose shape can be defined as linear, concave, or convex. Although only two test problems were instantiated, the toolkit provides a methodology for designing scalable test problems with Pareto optimal surfaces having an arbitrary number of vertices.

A Toolkit

A remarkable aspect of the Masuda-Nojima-Ishibuchi (MNI) test suite, is that at different distances from the PF, different PF shapes can be produced. This offers certain difficulty in identifying position and distance parameters.
Multi-Line Distance Minimization Problems

Li et al. [2018] proposed and discussed a class of scalable multi-objective test instances called multi-line distance minimization problems (ML-DMP) to evaluate the performance of evolutionary approaches in high-dimensional objective spaces. The test problems proposed in this test suite, were mainly introduced for visual examination of solution diversity in the decision space instead of the objective space.

Multi-Line Distance Minimization Problems

Two are the main characteristics of this test suite: 1) the Pareto optimal solutions lie in a regular polygon in a two-dimensional decision space, and 2) these solutions are similar (in the sense of Euclidean geometry) to their images in high-dimensional spaces. This allows to understand the distribution of the objective vector set by observing the solution set in the two-dimensional decision space in which these test problems are defined.
Ishibuchi et al. [2017] proposed minus versions of the DTLZ and WFG test problems (namely minus-DTLZ (DTLZ\(^{-1}\)) and minus-WFG (WFG\(^{-1}\)), respectively) as scalable test problems with clear differences from their original versions. These test problems stand out mainly because the Pareto optimal fronts of the original DTLZ and WFG test problems are inverted to obtain a similar effect as in the mirror LSS test problems [Emmerich, 2007].

Test Problems

Minus Problems

However, in this test suite, different geometries (the geometries used in the DTLZ and WFG test problems) are employed instead of being limited to the superspheres as in the case of the mirror LSS test problems.

Some important key points to consider are the following: 1) all the test problems maintain the same properties respect to the difficulties of the distance functions; and 2) different test problems promote the design of diversity mechanisms to achieve a proper representation of the inverted DTLZ and WFG Pareto optimal fronts.
MaF

Cheng et al. [2017] presented a compilation of 15 test problems which are presented as a scalable test suite, called MaF. In this test suite, the authors’ intention is to compile a set of test problems with different features in order to evaluate many-objective evolutionary approaches.

Most of the test problems included in this test suite were taken from already formulated test problems such as WFG, DTLZ, and ML-DMP, among other test suites. Thus, a wide variety of features can be found in this test suite which, indeed, shall be able to assess the robustness of many-objective evolutionary approaches.

Approaches to Design MOPs

In general, there are three different techniques which have been adopted in the construction of multi-objective test problems [Deb et al., 2005]: (1) Multiple single-objective approach, (2) Bottom-Up approach, and (3) Constraint surface approach.

Multiple Single-Objective Approach

This is an intuitive method that combines a number of single-objective optimization problems to formulate a multi-objective model. This strategy was extensively adopted in the early days of evolutionary multi-objective optimization research. The main disadvantage of this approach is that the **Pareto set** (PS) and the **Pareto front** (PF) are unknown, and depending of the single-objective functions, they can be very difficult to state. This, in fact, complicates the analysis of results and the comparison of MOEAs may become unfair. Nonetheless, this methodology has been recently adopted to formulate new multi-objective test problems.
Test Problems

Bottom-Up Approach

The bottom-up approach [Deb et al., 2005] is a flexible method that has facilitated the design of multi-objective test problems. In this approach, the \textit{Pareto optimal front}, the \textit{objective space} and the \textit{decision space} are separately constructed. Concretely, the decision variables are splitted into two groups: “\textit{position}” and “\textit{distance}” parameters.

The Pareto optimal surface is constructed by parametric functions (\textit{position functions}) whose inputs are the position parameters. The objective space is stated by constructing an extreme boundary surface parallel to the Pareto optimal surface, so that the hyper-volume bounded by these two surfaces constitutes the attainable objective space. Finally, each decision variable vector is mapped into objective space.
Bottom-Up Approach

This task is carried out by defining linear/nonlinear functions where the inputs are the distance parameters. Such functions (known as *distance functions*) establish the distance of the objective vectors to the PF. Therefore, the difficulty to approximate solutions to the PF depends directly on the difficulty of solving such distance functions.

Because of its flexibility, the bottom-up approach has been successfully employed in the construction of multi-objective test problems, particularly in the design of scalable test problems. However, most of the test suites adopting the bottom-up approach assume that position and distance parameters are completely uncorrelated—i.e. they can be easily identified—which is something hardly seen in real-world problems.
Constraint Surface Approach

This method was introduced to construct constrained multi-objective test problems [Deb et al., 2005]. Unlike the bottom-up approach that starts from a pre-defined Pareto optimal surface, the constraint surface approach first states the overall search space.

Second, a number of linear/non-linear constraints involving the objective function values is added, thus erasing part of the objective space (i.e., restricting the search space). Finally, by defining linear/non-linear objective functions, the decision variable space is mapped into the objective space.

Test Problems

Recommendations and Features

The construction of multi-objective test problems should satisfy some requirements and should include characteristics aimed to evaluate specific components of MOEAs. In particular, when a test instance possesses different characteristics, the test problem should evaluate the robustness of a MOEA, i.e., the capability of a MOEA to solve a test problem with a certain number of features.

Several criteria for the construction of multi-objective test instances have been discussed by a number of researchers, particularly in the pioneering works of Deb et al. [2005] and Huband et al. [2006].

Carlos A. Coello Coello
Multi-Objective Optimization
Huband et al. [2006] analyzed and justified different requirements which should be considered in the design of multi-objective test problems. We will show next the seven recommendations (R1–R7) and the five features (F1–F5) discussed by Huband et al. [2006]. However, because of the inherent progress on evolutionary multi-objective optimization, other features (F6–F8) are also added and described.
### Test Problems

#### Recommendations

<table>
<thead>
<tr>
<th>Recommendation</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>R1: No Extremal Parameters</td>
<td>Prevents exploitation by truncation based correction operators</td>
</tr>
<tr>
<td>R2: No Medial Parameters</td>
<td>Prevents exploitation by intermediate recombination</td>
</tr>
<tr>
<td>R3: Scalable Number of Parameters</td>
<td>Increases flexibility, demands scalability</td>
</tr>
<tr>
<td>R4: Scalable Number of Objectives</td>
<td>Increases flexibility, demands scalability</td>
</tr>
<tr>
<td>R5: Dissimilar Parameter Domains</td>
<td>Encourages EAs to scale mutation strengths appropriately</td>
</tr>
<tr>
<td>R6: Dissimilar Trade-off Ranges</td>
<td>Encourages normalization of objective values</td>
</tr>
<tr>
<td>R7: Pareto Optima Known</td>
<td>Facilitates the use of measures, analysis of results, in addition to other benefits</td>
</tr>
</tbody>
</table>

#### Features

<table>
<thead>
<tr>
<th>Feature</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>F1: Pareto Optimal Geometry</td>
<td>Convex, linear, concave, mixed, degenerate, disconnected, or some combination</td>
</tr>
<tr>
<td>F2: Parameter dependencies</td>
<td>Objectives can be separable or non-separable</td>
</tr>
<tr>
<td>F3: Bias</td>
<td>Substantially more solutions exist in some regions of fitness space than they do in others</td>
</tr>
<tr>
<td>F4: Many-to-one mappings</td>
<td>Pareto one-to-one/many-to-one, flat regions, isolated optima</td>
</tr>
<tr>
<td>F5: Modality</td>
<td>Uni-modal, or multi-modal (possibly deceptive multi-modality)</td>
</tr>
<tr>
<td>F6: Difficult Pareto Set Topology</td>
<td>Pareto set difficult to characterize</td>
</tr>
<tr>
<td>F7: Difficult Pareto Front Shape</td>
<td>Pareto optimal front difficult to estimate</td>
</tr>
<tr>
<td>F8: Correlation of Position and Distance Functions</td>
<td>Dependencies between position and distance functions</td>
</tr>
<tr>
<td>F9: Single Optimal Solution for a High Number of Objectives</td>
<td>Single objective solution for multiple objective functions</td>
</tr>
</tbody>
</table>
Limitations of Current Benchmarks

All modern benchmarks follow the bottom-up approach. This form to formulate multi-objective test problems splits the construction of the PF and the design of the search space which, in fact, facilitates the construction of multi-objective problems specially in high-dimensional objective spaces.

Recommendations (R1–R7) are partially covered by most of the modern test problems, being the WFG test suite, the only set of problems that satisfies entirely such requirements.

In the case of features related to the search space (F2–F5), most of the modern test problems do not adhere or cannot fit in a specific or desirable combination of features. While these features can be studied separately, there is no reason to assume that a real-world problem does not adhere simultaneously to several of these features at the same time.
Limitations of Current Benchmarks

Although one might doubt the existence of multi-objective problems having a combination of characteristics different from the ones formulated in the existing scalable test suites, according to the No-Free Lunch theorem, this overestimation does not hold.

In other words, there is an immense number of formulated and unformulated real-world problems and it is reasonable to think that any of them may have a wide variety of features not contemplated in any already formulated artificial test problem.

Thus, the inflexibility of configuring (in an easy way) scalable test problems with a desirable combination of features, becomes also a limitation of the existing scalable test suites.

On the other hand, difficult PS topologies (F6) are not considered by most of the modern test suites, which becomes a limitation.
Limitations of Current Benchmarks

An important issue to consider in scalable test problems refers to the shape of the Pareto optimal front. In this regard, the Pareto optimal fronts of the existing scalable test problems combine a variety of different geometries including convexity, concavity and/or linearity.

Several of the existing test problems (e.g., from the DTLZ and WFG test suites) can be characterized by an \((M - 1)\)-simplex. Test problems having this type of shapes are easy to solve for some evolutionary approaches.

In the specialized literature, we can find several MOPs in which their PF approximations draw strange geometries that do not follow exactly the shape of an \((M - 1)\)-simplex, see for example the problems presented in [Dirkx and Mooij, 2014].

Limitations of Current Benchmarks

Most of the modern test problems do not follow the property of difficult PF shape (F7) that has been suggested to evaluate diversity mechanisms in MOEAs. This, in fact, becomes a limitation of the constructed test problems and motivates to design new geometries different from those included in the state-of-the-art test suites.

Another important property that should be considered in the construction of scalable test problems is regarding the correlation between position and distance functions (F8). Most of the modern test problems do not follow this property which complicates the identification of position and distance parameters.

Although there exist approaches employed to correlate position and distance functions (e.g. the modular approach), the investigation and development of a more flexible design approach for constructing scalable test problems—where position and distance variables are indistinguishable and the true PS and PF can be analytically known—is in fact a good path for future research.