Multi-Objective Optimization Using Metaheuristics

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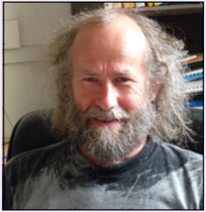
LECTURE 4

Test Problems



Some examples of test suites that have been proposed in the specialized literature to evaluate single-objective evolutionary algorithms are the following:

- The 5 test problems from De Jong [1975] for unconstrained optimization.
- The 12 test problems from Michalewicz & Schoenauer [1996] for constrained optimization.
- The 62 test problems from Schwefel [1995] for evaluating evolution strategies.

Test Problems

- The test problems proposed by Whitley et al. [1996] and Goldberg [1989].
- The test problems from Yao & Liu [1996; 1997] used to assess performance of evolutionary programming and evolution strategies.
- The deceptive problems from Goldberg and Mühlenbein.
- The 8 test problemas from Digalakis & Margaritis [2000].
- The multimodal test problems from Levy [1981], the test problems from Corana [1987], the test problems from Freudenstein-Roth and the test problems from Goldstein-Price [1981].
- Ackley's function and Wirestrass' function [Bäck et al., 1997].

Test Problems

A set of test problems to evaluate the performance of a MOEA should include the following features (both in genotypic and in phenotypic space):

- Continuous vs. discontinuous vs. discrete
- Differentiable vs. non-differentiable
- Convex vs. concave
- Modality (unimodal, multi-modal)
- Numerical vs. alphanumeric
- Quadratic vs. nonquadratic
- Type of constraints (equalities, inequalities, linear, nonlinear)
- Low vs. high dimensionality (genotype, phenotype)
- Deceptive vs. nondeceptive
- Biased vs. unbiased portions of the true Pareto front

Test Problems

Test problems should range in difficulty from “easy” to “hard” as well as attempt to represent generic real-world situations.

Dynamically changing environments can include “moving cones” [Morrison & de Jong, 1999] with movement ranging from predictable to chaotic to non-stationary and deceptive.

R.W. Morrison and K.A. de Jong, “**A Test Problem Generator for Non-Stationary Environments**”, in *1999 IEEE Congress on Evolutionary Computation*, pp. 2047–2053, IEEE Press, Washington, D.C., USA, 1999.

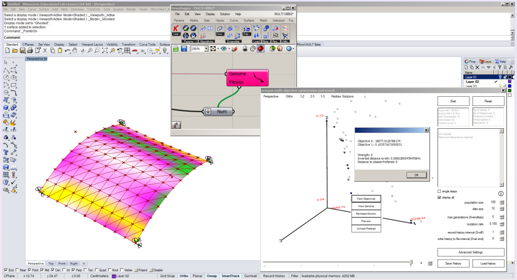
Test Problems

One should also consider the following guidelines suggested by Whitley et al. [1996] in developing generic test suites:

- Some test suite problems are *resistant* to simple search strategies.
- Test suites contain nonlinear, unseparable & unsymmetric problems.
- Test suites contain scalable problems.
- Some test suite problems have scalable evaluation cost.
- Test problems have a canonical representation (ease of use).

D. Whitley, K. Mathias, S. Rana and J. Dzubera, "**Evaluating Evolutionary Algorithms**", *Artificial Intelligence*, **85**:245–276, 1996.

Test Problems



Ideally, test problems used to evaluate a MOEA should contain features and difficulties similar to those found in the real-world problem(s) that we aim to solve.

However, the specialized literature presents a wide number of “artificial” test problems that emphasize certain aspects that are indeed difficult for most MOEAs, but that don’t necessarily represent the difficulties found in real-world problems.

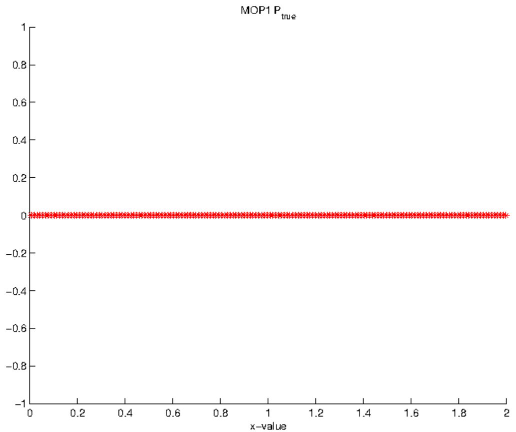
Unconstrained Problems

MOP 1: This is the first test problem used by David Schaffer. Historically, it has a very high relevance, because it was the first test problem proposed to evaluate the performance of a MOEA. However, this problem is so simple that its Pareto front can be obtained in an analytic form. PF_{true} is convex and the problem has a single decision variable.

$F = (f_1(x), f_2(x))$, where

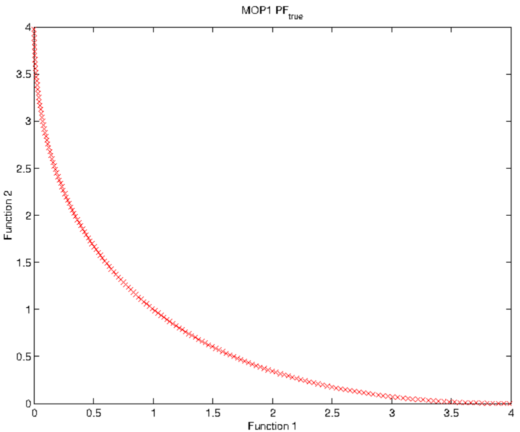
$$\begin{aligned} f_1(x) &= x^2, \\ f_2(x) &= (x-2)^2 \end{aligned}$$

where: $-10^5 \leq x \leq 10^5$



Unconstrained Problems

P_{true} of **MOP 1**



Unconstrained Problems

PF_{true} of **MOP 1**

Test Problems

Unconstrained Problems

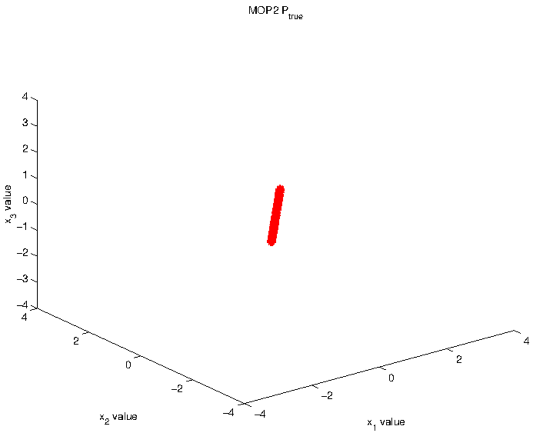
MOP 2: This is the second test problem proposed by Fonseca. It is scalable. It is possible to add decision variables to this test problem without changing the shape of PF_{true} (the Pareto front is concave in this case).

$F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$f_1(\vec{x}) = 1 - \exp(-\sum_{i=1}^{n}(x_i - \frac{1}{\sqrt{n}})^2),$$

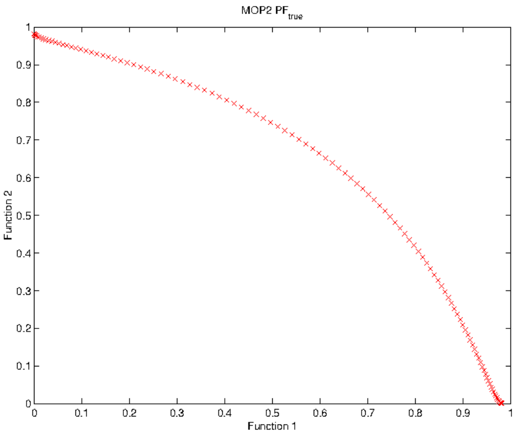
$$f_2(\vec{x}) = 1 - \exp(-\sum_{i+1}^{n}(x_i + \frac{1}{\sqrt{n}})^2)$$

where: $-4 \leq x_i \leq 4;\ i = 1, 2, 3$



Unconstrained Problems

P_{true} of **MOP 2**



Unconstrained Problems

PF_{true} of **MOP 2**

Unconstrained Problems

MOP 3: Proposed by Carlo Poloni. Both P_{true} and PF_{true} are disconnected.

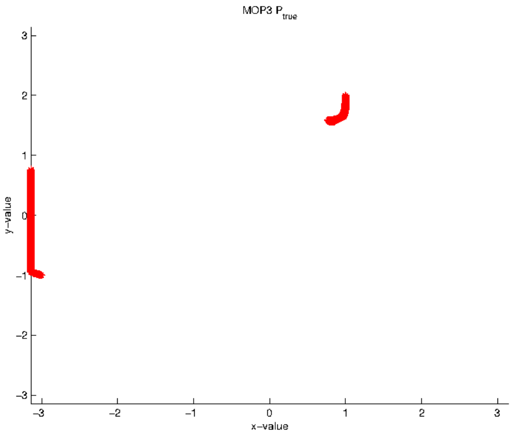
Maximize $F = (f_1(x, y), f_2(x, y))$, where

$$\begin{aligned} f_1(x, y) &= -[1 + (A_1 - B_1)^2 + (A_2 - B_2)^2], \\ f_2(x, y) &= -[(x + 3)^2 + (y + 1)^2] \end{aligned}$$

where: $-3.1416 \leq x, y \leq 3.1416$,

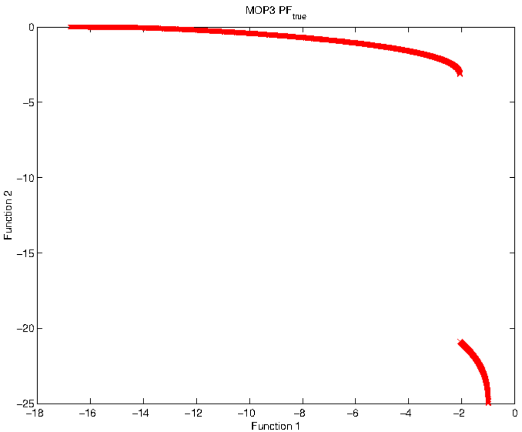
$$\begin{aligned} A_1 &= 0.5 \sin 1 - 2 \cos 1 + \sin 2 - 1.5 \cos 2, \\ A_2 &= 1.5 \sin 1 - \cos 1 + 2 \sin 2 - 0.5 \cos 2, \\ B_1 &= 0.5 \sin x - 2 \cos x + \sin y - 1.5 \cos y, \\ B_2 &= 1.5 \sin x - \cos x + 2 \sin y - 0.5 \cos y \end{aligned}$$

Test Problems



Unconstrained Problems

P_{true} of **MOP 3**



Unconstrained Problems

PF_{true} of **MOP 3**

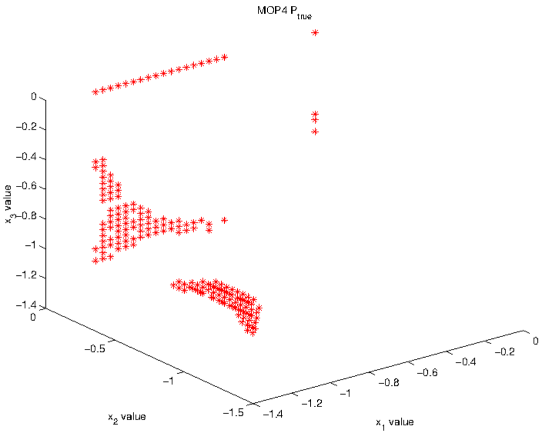
Unconstrained Problems

MOP 4: Proposed by Kursawe. There are disconnected and asymmetrical portions in P_{true}. PF_{true} consists of 3 disconnected curves. It allows the use of an arbitrary number of decision variables, although scaling this test problem changes the shape of PF_{true}.

$F = (f_1(\vec{x}), f_2(\vec{x}))$, where

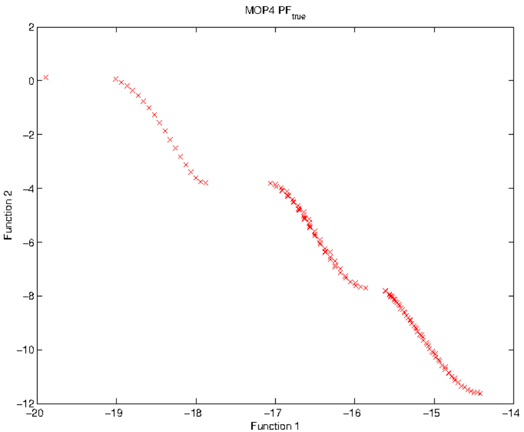
$$\begin{aligned} f_1(\vec{x}) &= \sum_{i=1}^{n-1}(-10e^{(-0.2)*\sqrt{x_i^2+x_{i+1}^2}}), \\ f_2(\vec{x}) &= \sum_{i=1}^{n}(|x_i|^a + 5\sin(x_i)^b) \end{aligned}$$

where: $-5 \leq x_i \leq 5$; $i = 1, 2, 3$; $a = 0.8, b = 3$



Unconstrained Problems

P_{true} of **MOP 4**



Unconstrained Problems

PF_{true} of **MOP 4**

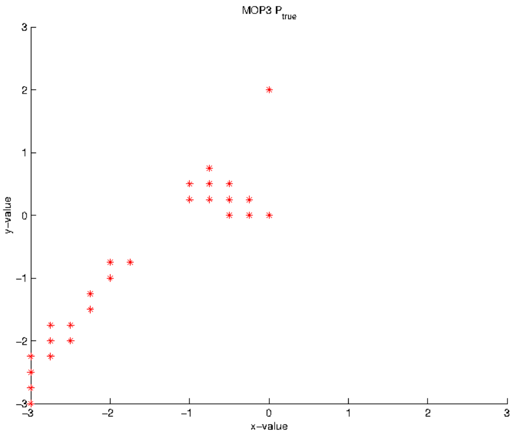
Unconstrained Problems

MOP 5: Proposed by Viennet. It has disconnected regions in P_{true}. PF_{true} is a three-dimensional curve.

$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where

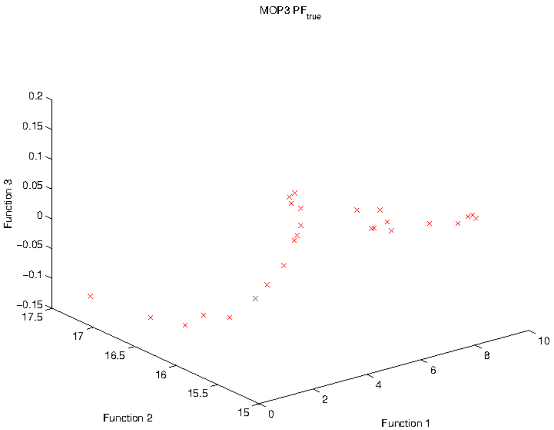
$$
\begin{aligned}
f_1(x,y) &= 0.5 * (x^2 + y^2) + \sin(x^2 + y^2), \\
f_2(x,y) &= \frac{(3x - 2y + 4)^2}{8} + \frac{(x - y + 1)^2}{27} + 15, \\
f_3(x,y) &= \frac{1}{(x^2 + y^2 + 1)} - 1.1e^{(-x^2 - y^2)}
\end{aligned}
$$

where: $-30 \leq x, y \leq 30$



Unconstrained Problems

P_{true} of **MOP 5**



Unconstrained Problems

PF_{true} of **MOP 5**

Unconstrained Problems

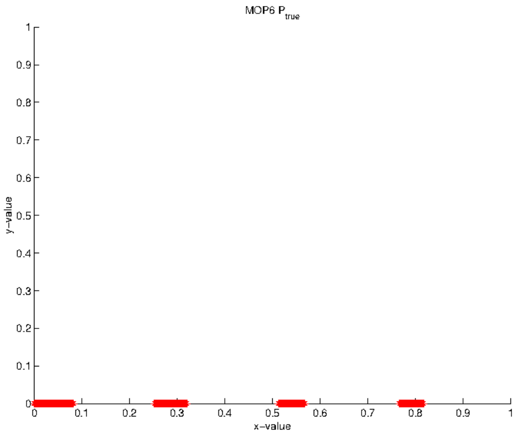
MOP 6: Proposed by Deb. Both P_{true} and PF_{true} are disconnected.

$F = (f_1(x,y), f_2(x,y))$, where

$$
\begin{aligned}
f_1(x,y) &= x, \\
f_2(x,y) &= (1+10y) * \\
&\quad [1 - (\frac{x}{1+10y})^{\alpha} - \frac{x}{1+10y}\sin(2\pi q x)]
\end{aligned}
$$

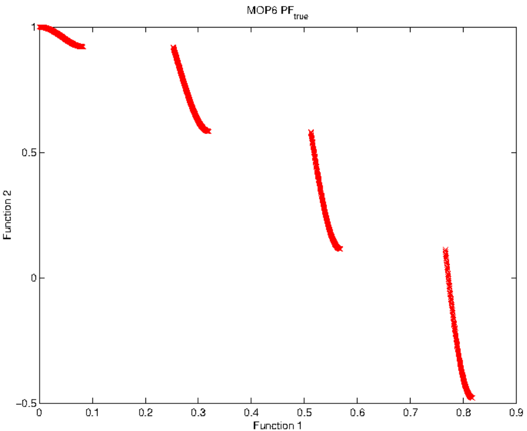
where: $0 \leq x, y \leq 1$,

$$
\begin{aligned}
q &= 4, \\
\alpha &= 2
\end{aligned}
$$



Unconstrained Problems

P_{true} of **MOP 6**



Unconstrained Problems

PF_{true} of **MOP 6**

Test Problems

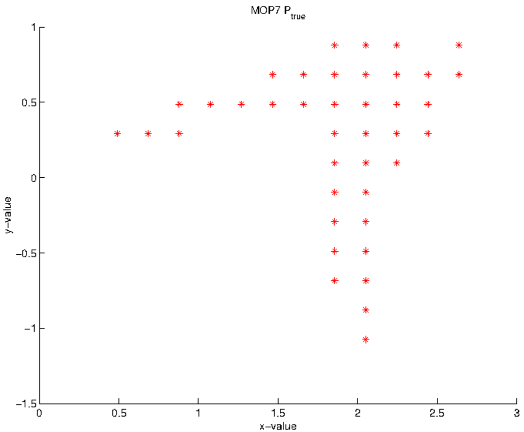
Unconstrained Problems

MOP 7: Proposed by Viennet. P_{true} is connected and PF_{true} is a surface. This problem is relatively easy to solve by any MOEA.

$F = (f_1(x,y), f_2(x,y), f_3(x,y))$, where

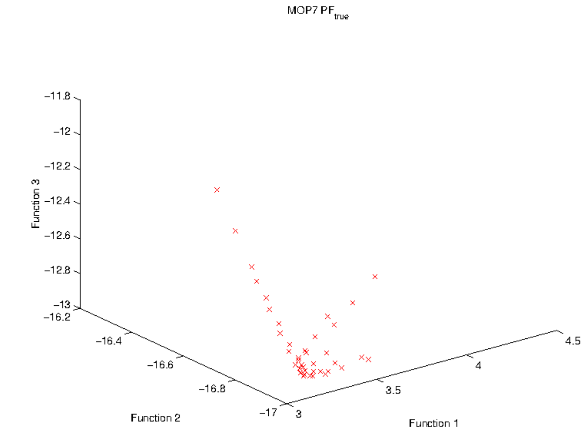
$$
\begin{aligned}
f_1(x,y) &= \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3, \\
f_2(x,y) &= \frac{(x+y-3)^2}{36} + \frac{(-x+y+2)^2}{8} - 17, \\
f_3(x,y) &= \frac{(x+2y-1)^2}{175} + \frac{(2y-x)^2}{17} - 13
\end{aligned}
$$

where: $-400 \leq x, y \leq 400$



Unconstrained Problems

P_{true} of **MOP 7**



Unconstrained Problems

PF_{true} of **MOP 7**

Constrained Problems

Historically, constraints have been handled in MOEAs through the use of penalty functions [Richardson et al., 1989].

However, many other methods to handle constraints are currently available, although few of them have been specifically designed for MOEAs.

Jon T. Richardson, Mark R. Palmer, Gunar Liepins, and Mike Hilliard, “**Some Guidelines for Genetic Algorithms with Penalty Functions**”, in J. David Schaffer (Ed), *Proceedings of the Third International Conference on Genetic Algorithms*, pp. 191–197, Morgan Kaufmann Publishers, San Mateo, California, USA, 1989.

B. Y. Qu and P. N. Suganthan, “**Constrained Multi-objective Optimization Algorithm with an Ensemble of Constraint Handling Methods**, *Engineering Optimization*, Vol. 43, No. 4, pp. 403–416, 2011.

Constrained Problems

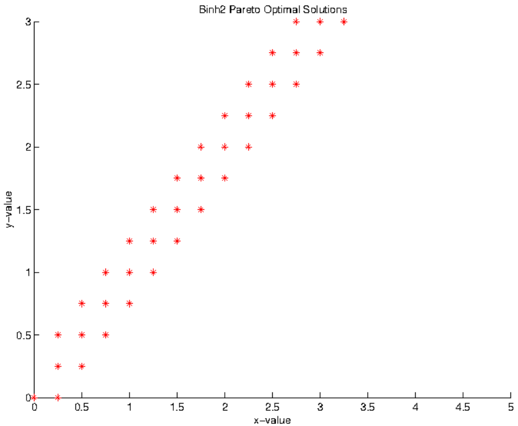
MOP-C1: Proposed by Binh. In this case, P_{true} is an area and PF_{true} is a single convex curve.

$F = (f_1(x, y), f_2(x, y))$, where

$$\begin{aligned} f_1(x, y) &= 4x^2 + 4y^2, \\ f_2(x, y) &= (x - 5)^2 + (y - 5)^2 \end{aligned}$$

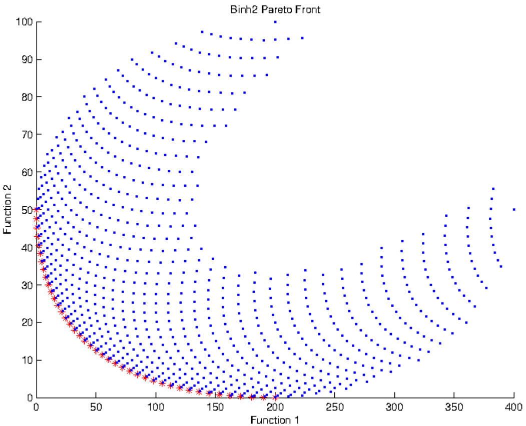
where:

$$0 \le x \le 5, \; 0 \le y \le 3$$
$$0 \ge (x - 5)^2 + y^2 - 25,$$
$$0 \ge -(x - 8)^2 - (y + 3)^2 + 7.7$$



Constrained Problems

P_{true} of **MOP-C1**



Constrained Problems

PF_{true} of **MOP-C1**

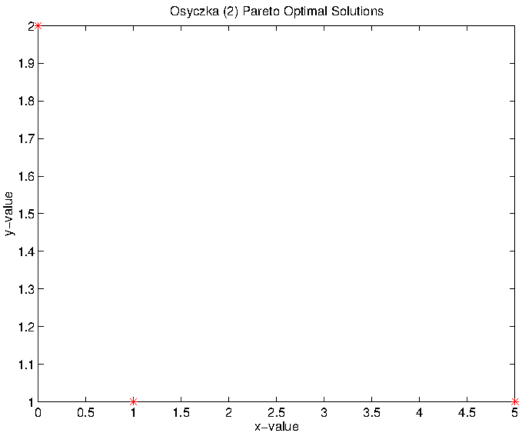
Constrained Problems

MOP-C2: Proposed by Osyczka. Both P_{true} and PF_{true} are disconnected.

$$
\begin{aligned}
f_1(\vec{x}) &= -(25(x_1-2)^2+(x_2-2)^2+(x_3-1)^2 \\
&\quad +(x_4-4)^2+(x_5-1)^2, \\
f_2(\vec{x}) &= x_1^2+x_2^2+x_3^2+x_4^2+x_5^2+x_6^2
\end{aligned}
$$

$0 \le x_1, x_2, x_6 \le 10$, $1 \le x_3, x_5 \le 5$, $0 \le x_4 \le 6$,

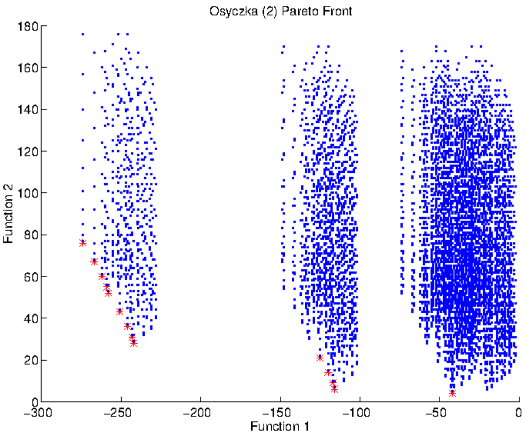
$$
\begin{aligned}
0 &\le x_1+x_2-2, \\
0 &\le 6-x_1-x_2, \\
0 &\le 2-x_2+x_1, \\
0 &\le 2-x_1+3x_2, \\
0 &\le 4-(x_3-3)^2-x_4 \\
0 &\le (x_5-3)^2+x_6-4
\end{aligned}
$$



Constrained Problems

P_{true} of **MOP-C2**

Test Problems



Constrained Problems

PF_{true} of **MOP-C2**

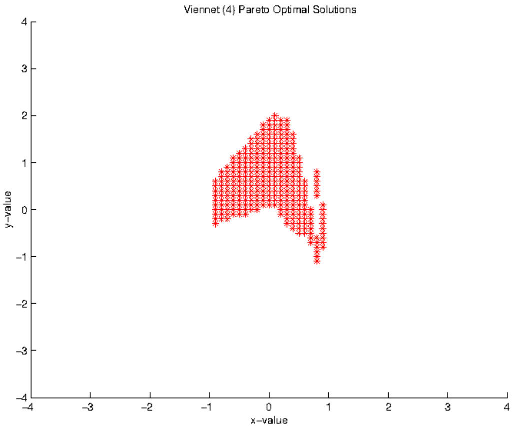
Constrained Problems

MOP-C3: Proposed by Viennet. P_{true} is connected but it's asymmetrical. PF_{true} is a 3D curve.

$$
\begin{aligned}
f_1(x,y) &= \frac{(x-2)^2}{2} + \frac{(y+1)^2}{13} + 3, \\
f_2(x,y) &= \frac{(x+y-3)^2}{175} + \frac{(2y-x)^2}{17} - 13, \\
f_3(x,y) &= \frac{(3x-2y+4)^2}{8} + \frac{(x-y+1)^2}{27} + 15
\end{aligned}
$$

$-4 \leq x, y \leq 4,$

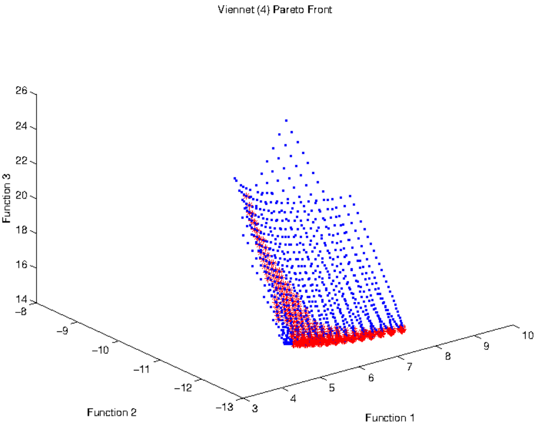
$$
\begin{aligned}
y &< -4x+4, \\
x &> -1, \\
y &> x-2
\end{aligned}
$$



Constrained Problems

P_{true} of **MOP-C3**

Test Problems



Constrained Problems

PF_{true} of **MOP-C3**

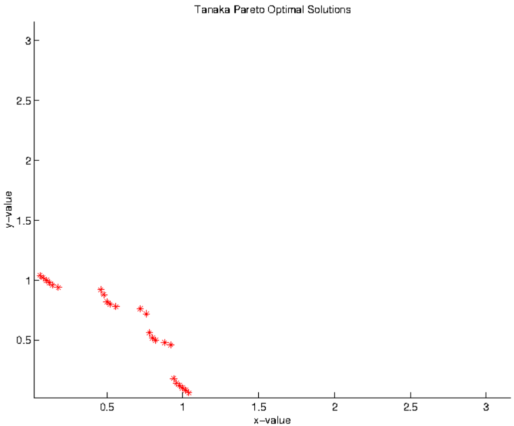
Constrained Problems

MOP-C4: Proposed by Tanaka. P_{true} is connected, but PF_{true} is disconnected.

$$
\begin{aligned}
f_1(x,y) &= x, \\
f_2(x,y) &= y
\end{aligned}
$$

$0 < x, y \le \pi,$

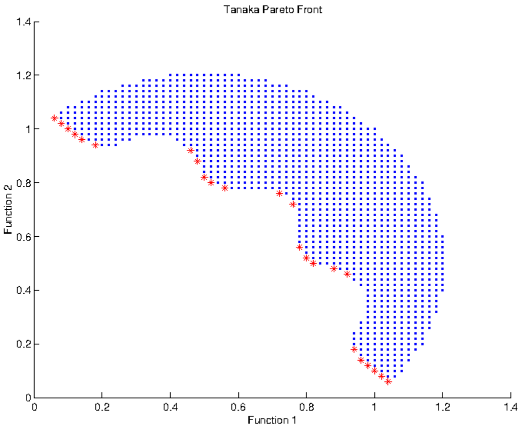
$$
\begin{aligned}
0 &\ge -(x^2) - (y^2) \\
&\quad +1 + \\
&\quad (a \cos \\
&\quad (b \arctan(x/y))) \\
a &= 0.1 \\
b &= 16
\end{aligned}
$$



Constrained Problems

P_{true} of **MOP-C4**

Test Problems



Constrained Problems

PF_{true} of **MOP-C4**

Test Problems Generators

MOP test functions can also be generated by using the single-objective functions. A methodology for constructing MOPs exhibiting desired characteristics has been proposed by Deb [1999].

Kalyanmoy Deb, "**Multi-Objective Genetic Algorithms: Problem Difficulties and Construction of Test Problems**", *Evolutionary Computation*, **7**(3):205-230, Fall 1999.

He points out that when computationally derived a non-uniform distribution of vectors may exist in some Pareto front. He limits his initial test construction efforts to unconstrained MOPs of only two functions; his construction methodology then places restrictions on the two component functions so that resultant MOPs exhibit desired properties. To accomplish this he defines various generic bi-objective optimization problems, such as the example of the next slide.

Test Problems

Test Problems Generators

Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= f(x_1, \ldots, x_m), \\ f_2(\vec{x}) &= g(x_{m+1}, \ldots, x_N)\, h(f(x_1, \ldots, x_m), g(x_{m+1}, \ldots, x_N)) \end{aligned} \quad (1)$$

where function f_1 is a function of $(m < N)$ decision variables and f_2 a function of all N decision variables.

The function g is one of $(N - m)$ decision variables which are not included in function f.

The function h is directly a function of f and g function values. The f and g functions are also restricted to positive values in the search space, i.e., $f > 0$ and $g > 0$.

Test Problems Generators

Deb lists five functions each for possible f and g instantiation, and four for h. These functions may then be "mixed and matched" to create MOPs with desired characteristics.

He states these functions have the following general effect:

- f – This function controls vector representation uniformity along the Pareto front.
- g – This function controls the resulting MOP's characteristics – whether it is multifrontal or has an isolated optimum.
- h – This function controls the resulting Pareto front's characteristics (e.g., convex, disconnected, etc.)

These functions respectively influence search along and towards the Pareto front, and the shape of a Pareto front in $\mathbb{R}^2$. Deb implies that a MOEA has difficulty finding PF_{true} because it gets "trapped" in a local Pareto front.

Test Problems Generators

MOP-G1: This is an example of the test problems generated with Deb's methodology. In this case, PF_{true} is convex.

$$
\begin{aligned}
f_1(x_1) &= x_1, \\
f_2(\vec{x}) &= g(1 - \sqrt{(f_1/g)}) \\
g(\vec{x}) &= 1 + 9\sum_{i=2}^{m} x_i/(m-1)
\end{aligned}
$$

$m = 30; 0 \leq x_i \leq 1$

Test Problems Generators

For constrained test MOPs, Deb [2001] suggests extending his methodology in the following way:

$$\begin{aligned} f_1(\vec{x}) &= x_1 \\ f_2(\vec{x}) &= g(\vec{x}) \exp(-f_1(\vec{x})/g(\vec{x})) \end{aligned}$$

subject to:

$$c_j(x) = f_2(\vec{x}) - a_j \exp(-b_j f_1(\vec{x})) \geq 0, \; j = 1, 2, ...J \tag{2}$$

There are J inequalities, each of which has 2 parameters (a_j, b_j), which makes that part of feasible region of the original (unconstrained) problem is now infeasible.

Test Problems

Test Problems Generators

Kalyanmoy Deb, Amrit Pratap and T. Meyarivan, "**Constrained Test Problems for Multi-objective Evolutionary Optimization**", in Eckart Zitzler et al. (Eds.), *First International Conference on Evolutionary Multi-Criterion Optimization*, pp. 284–298. Springer-Verlag. Lecture Notes in Computer Science No. 1993, 2001

An example of this methodology is the following:
Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= x_1 \\ f_2(\vec{x}) &= (1 + x_2)/x_1 \end{aligned}$$

$$0.1 \leq x_1 \leq 1.0$$

$$0.0 \leq x_2 \leq 5.0$$

subject to:

$$\begin{aligned} c_1(\vec{x}) &= x_2 + 9x_1 \geq 6 \\ c_2(\vec{x}) &= -x_2 + 9x_1 \geq 1 \end{aligned} \tag{3}$$

Test Problems Generators

In fact, the next generic form is suggested:
Minimize $F = (f_1(\vec{x}), f_2(\vec{x}))$, where

$$\begin{aligned} f_1(\vec{x}) &= x_1 \\ f_2(\vec{x}) &= g(\vec{x})(1 - f_1(\vec{x})/g(\vec{x}) \end{aligned}$$

subject to:

$$\begin{aligned} c_j(\vec{x}) = \; & \cos(\theta)(f_2(\vec{x}) - e) - \sin(\theta)f_1(\vec{x})) \geq \\ & a|\sin(b\pi(\sin)\theta)(f_2(\vec{x}) - e) + \cos(\theta)f_1(\vec{x}))^c)|^d, \\ & j = 1, 2, ...J \end{aligned} \tag{4}$$

Test Problems Generators

With 6 parameters (θ, a, a, c, d, e), x_1 is restricted to the range [0,1] and $g(\vec{x})$ determines the bounds of the other decision variables.

Selecting values for the 6 parameters, we can generate different fitness landscapes.

It is worth noting that d controls the length of the continuous region of the Pareto front. As we decrease this region, a MOEA will tend to find less points of PF_{true} because of the discretization of $\vec{x}$.

Test Problems Generators

If we increase the value of a, the length of the "cuts" becomes more profound, which requires the search to proceed through a narrowed corridor. Evidently, this makes more difficult the search.

We can also depart from the periodic disconnected regions of PF_{true} by changing c from its initial value of 1.

θ and e control the slope and the change of direction of PF_{true}, respectively.

Zitzler-Deb-Thiele (ZDT) Test Problems

Each of the test problems shown next is structured in the same way and it consists of 3 functions f_1, g, h:

$$\begin{aligned} \textit{Minimize}: F(\vec{x}) &= (f_1, f_2), \\ \textit{subject to}: f_2(\vec{x}) &= g(x_2, \ldots, x_m)h(f_1(x_1), g(x_2, \ldots, x_m)), \\ \textit{where}: \vec{x} &= (x_1, \ldots, x_M). \end{aligned} \tag{5}$$

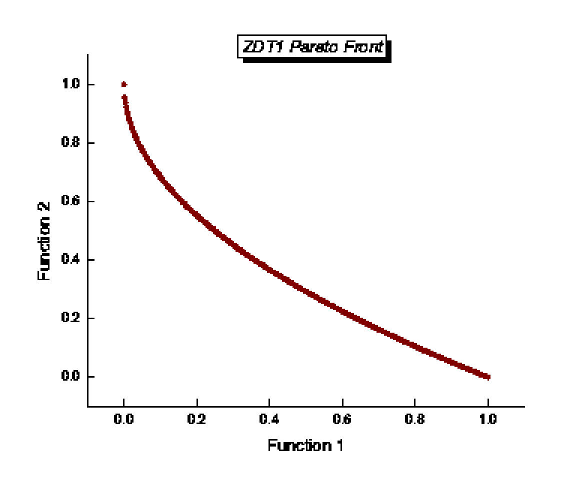
f_1 is a function of only the first decision variable, g is a function of the $m-1$ remaining decision variables, and the parameters of h are the values of f_1 and g.

Zitzler-Deb-Thiele (ZDT) Test Problems

Eckart Zitzler, Kalyanmoy Deb and Lothar Thiele, "**Comparison of Multiobjective Evolutionary Algorithms: Empirical Result**", *Evolutionary Computation*, **8**(2):173-195, Summer 2000.

The test problems differ in these 3 functions and in the number of decision variables m, as well as in the values that the decision variables can take. These problems have been heavily used to validated MOEAs in the specialized literature.

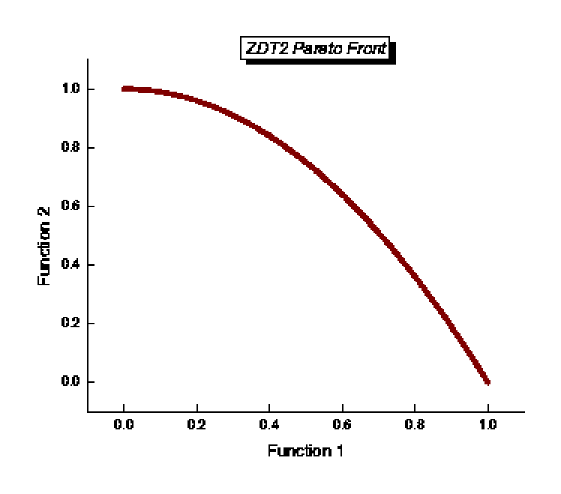
Test Problems



ZDT Test Problems

PF_{true} of **ZDT1**

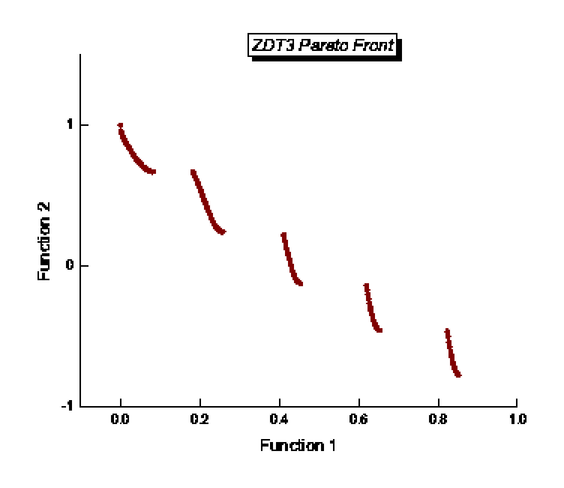
Test Problems



ZDT Test Problems

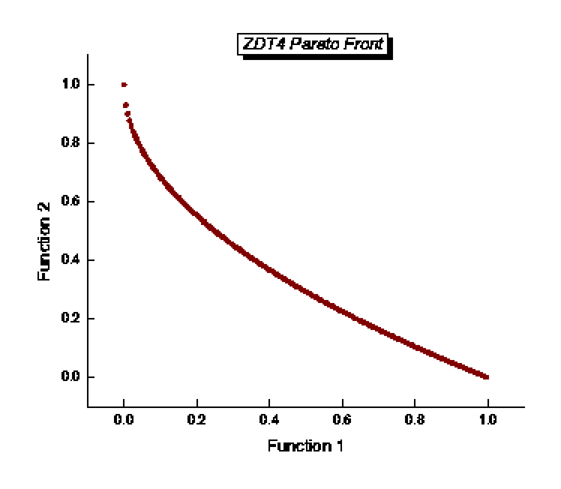
PF_{true} of **ZDT2**

Test Problems



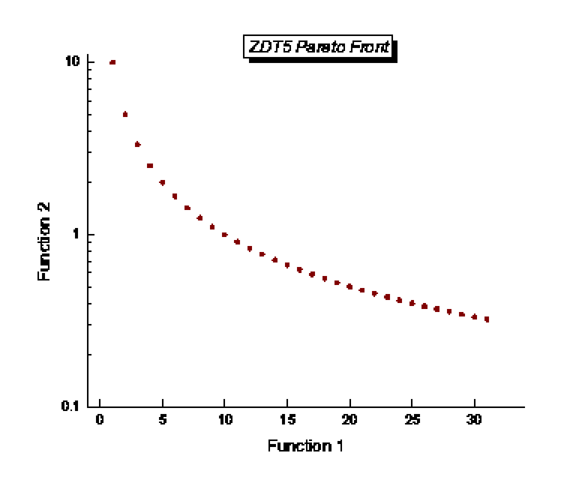
ZDT Test Problems

PF_{true} of **ZDT3**



ZDT Test Problems

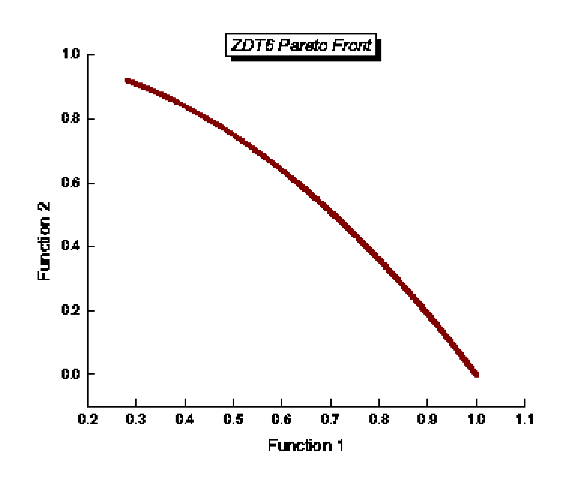
PF_{true} of **ZDT4**



ZDT Test Problems

PF_{true} of **ZDT5**

Test Problems



ZDT Test Problems

PF_{true} of **ZDT6**

Test Problems

Deb-Thiele-Laumanns-Zitzler (DTLZ) Test Problems

Another desirable feature of a test problem is that it can scale up to any number of dimensions.

Since the mapping between the genotypic and the phenotypic space can be considerably nonlinear, we can exploit this property to generate test problems with a high degree of difficulty.

Deb et al. [2002,2005] proposed the so-called Deb-Thiele-Laumanns-Zitzler (DTLZ) test suite in which the problems are scalable to a number of objectives defined by the user. This test suite has also been very popular in the specialized literature.

Kalyanmoy Deb, Lothar Thiele, Marco Laumanns and Eckart Zitzler, "**Scalable Test Problems for Evolutionary Multiobjective Optimization**", in Ajith Abraham, Lakhmi Jain and Robert Goldberg (editors), *Evolutionary Multiobjective Optimization. Theoretical Advances and Applications*, pp. 105–145, Springer, USA, 2005.

DTLZ1

PF_{true} is linear, separable and multimodal.

Minimize:

$$f_1(x) = \frac{1}{2}x_1 x_2 \ldots x_{M-1}(1 + g(x_M)), \tag{6}$$

$$f_2(x) = \frac{1}{2}x_1 x_2 \ldots (1 - x_{M-1})(1 + g(x_M)), \tag{7}$$

$$\vdots \qquad \vdots \tag{8}$$

$$f_{M-1}(x) = \frac{1}{2}x_1(1 - x_2)(1 + g(x_M)), \tag{9}$$

$$f_M(x) = \frac{1}{2}(1 - x_1)(1 + g(x_M)), \tag{10}$$

$$\text{subject to } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n \tag{11}$$

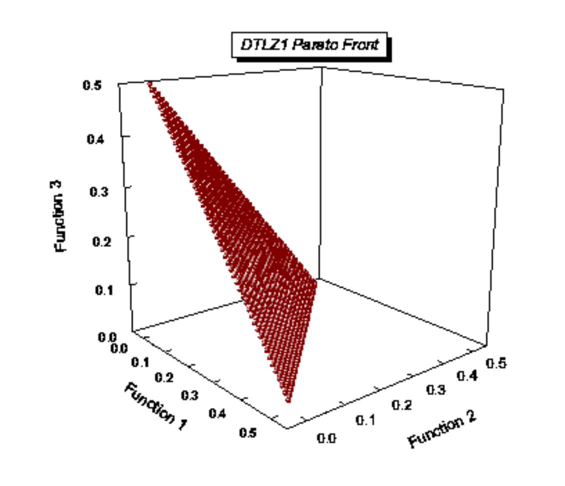
$$\text{where: } g(x_M) = 100\left[|x_M| + \sum_{x_i \in x_M}(x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))\right] \tag{12}$$

DTLZ1

It is normally adopted with $M = 3$. The Pareto optimal set is located at $x_M^* = 0$ and the values of the objective functions at the linear hyperplane $\sum_{m=1}^{M} = 0.5$.

The search space contains $(11^k - 1)$ local Pareto fronts (k is a value defined by the user, such that the number of decision variables is: $n = M + k - 1$. It is common to adopt $k = 5$).

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ1**

DTLZ2

Minimize:

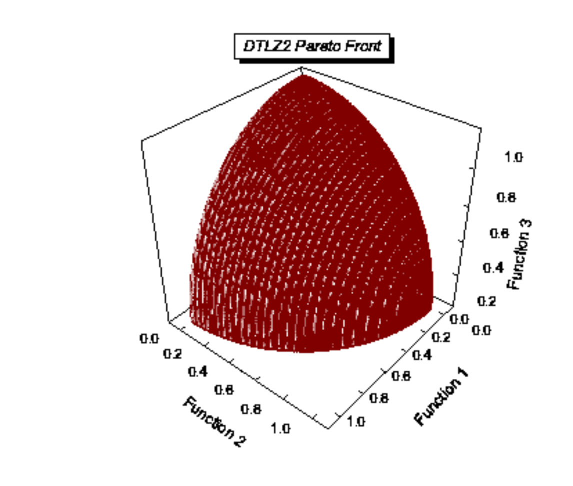
$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\dots\cos(x_{M-2}\pi/2)\cos(x_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\dots\cos(x_{M-2}\pi/2)\sin(x_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\dots\sin(x_{M-2}\pi/2),\\
\vdots & \qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(x_1\pi/2)\sin(x_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(x_1\pi/2).\\
\text{subject to: } & 0 \le x_i \le 1 \quad \forall \quad i = 1, 2, \dots, n\\
\text{where: } g(x_M) &= \sum_{x_i \in X_M} (x_i - 0.5)^2
\end{aligned}
$$

Test Problems

DTLZ2

The Pareto optimal set is located at: $x_i = 0.5$ for every $x_i \in x_M$ and all the objective functions have to satisfy: $\sum_{i=1}^{M}(f_i)^2 = 1$. It is suggested to use $k = |x_M| = 10$.

The total number of decision variables is: $n = M + k - 1$.



DTLZ Test Problems

PF_{true} of **DTLZ2**

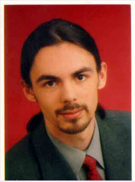
DTLZ3

Minimize:

$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\cos(x_{M-2}\pi/2)\cos(x_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\cos(x_{M-2}\pi/2)\sin(x_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(x_1\pi/2)\cos(x_2\pi/2)\ldots\sin(x_{M-2}\pi/2),\\
&\vdots \qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(x_1\pi/2)\sin(x_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(x_1\pi/2).
\end{aligned}
$$

$$\text{subject to: } 0 \le x_i \le 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } g(x_M) = 100[|x_M| + \sum_{x_i \in x_M} (x_i - 0.5)^2 - \cos(20\pi(x_i - 0.5))]$$



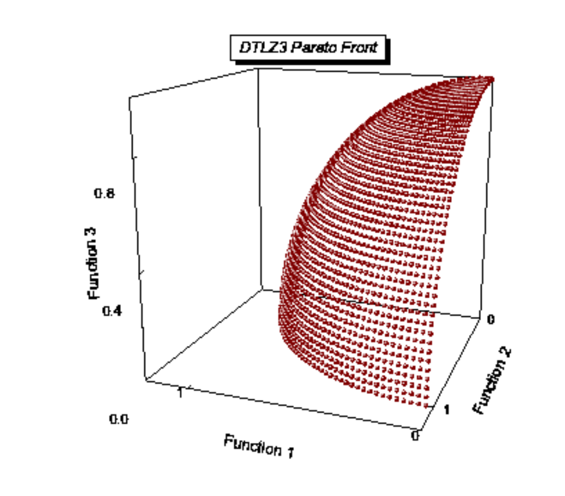
DTLZ3

It is suggested that $k = |x_M| = 10$. There is a total of $n = M + k - 1$ decision variables.

The function g described before, introduces ($3k$ - 1) false Pareto fronts. All of these false Pareto fronts are parallel to the global Pareto front and, therefore, a MOEA can get easily trapped in one of them before converging to the Pareto optimal front which is located at $g^* = 0$.

The true Pareto front corresponds to $x_M = (0.5, \ldots, 0.5)^T$.

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ3**

Test Problems

DTLZ4

Minimize:

$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\cos(x_2^{\pi}\pi/2)\ldots\cos(x_{M-2}^{\pi}\pi/2)\cos(x_{M-1}^{\pi}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\cos(x_2^{\pi}\pi/2)\ldots\cos(x_{M-2}^{\pi}\pi/2)\sin(x_{M-1}^{\pi}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\cos(x_2^{\pi}\pi/2)\ldots\sin(x_{M-2}^{\pi}\pi/2),\\
\vdots &\qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(x_1^{\pi}\pi/2)\sin(x_2^{\pi}\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(x_1^{\pi}\pi/2).
\end{aligned}
$$

subject to: $0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$

where: $g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2$

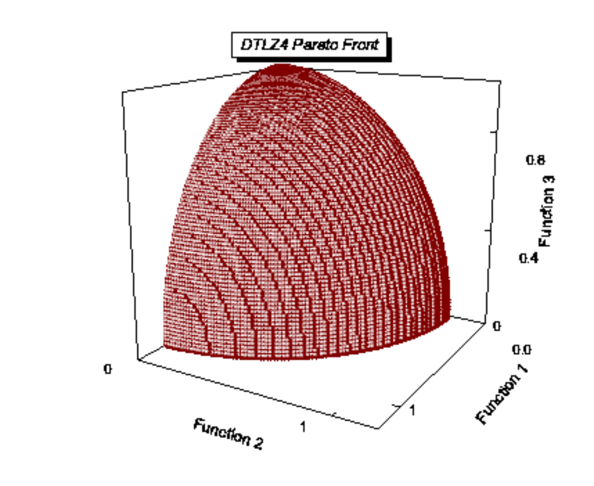
DTLZ4

It is suggested to use $\alpha = 100$ in this case. Again, all the decision variables x_1 to x_{M-1} are varied in the range $(0:1)$.

It is also suggested to use $k = 10$. There are $n = M + k - 1$ decision variables in this problem.

In this case, there is dense set of solutions close to the plane $f_M - f_1$.

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ4**

DTLZ5

Minimize:

$$
\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\dots\cos(\theta_{M-2}\pi/2)\cos(\theta_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\dots\cos(\theta_{M-2}\pi/2)\sin(\theta_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\dots\sin(\theta_{M-2}\pi/2),\\
&\vdots \qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\sin(\theta_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(\theta_1\pi/2).
\end{aligned}
$$

$$\text{subject to: } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } \theta_i = \frac{\pi}{4(1+g(x_M))}(1+2g(x_M)x_i), \text{for } i = 2, 3, \dots, (M-1)$$

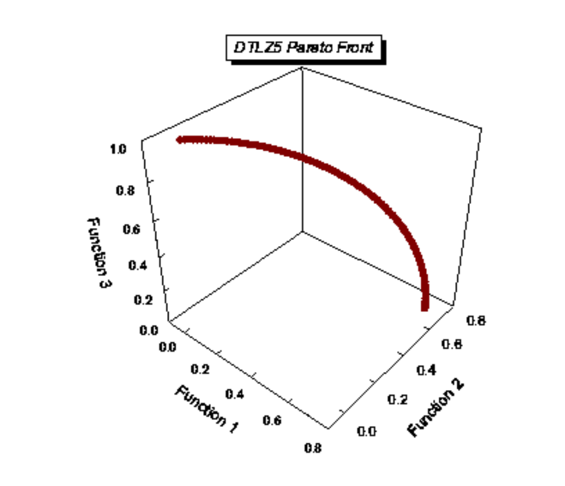
$$g(x_M) = \sum_{x_i \in X_M} (x_i - 0.5)^2$$

DTLZ5

It is suggested to use function g with $k = |x_M| = 10$. Also, there are $n = M + k - 1$ decision variables and the Pareto optimal set corresponds to $x_i = 0.5$ for every $x_i \in x_M$ and every objective function must satisfy: $\sum_{i=1}^{M} (f_i)^2 = 1$.

This problem evaluates the capability of a MOEA to converge to a curve.

It is suggested to use ($M \in [5, 10]$).



DTLZ Test Problems

PF_{true} of **DTLZ5**

DTLZ6

Minimize:

$$\begin{aligned}
f_1(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\ldots\cos(\theta_{M-2}\pi/2)\cos(\theta_{M-1}\pi/2),\\
f_2(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\ldots\cos(\theta_{M-2}\pi/2)\sin(\theta_{M-1}\pi/2),\\
f_3(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\cos(\theta_2\pi/2)\ldots\sin(\theta_{M-2}\pi/2),\\
&\vdots \qquad \vdots\\
f_{M-1}(x) &= (1+g(x_M))\cos(\theta_1\pi/2)\sin(\theta_2\pi/2),\\
f_M(x) &= (1+g(x_M))\sin(\theta_1\pi/2).
\end{aligned}$$

$$\text{subject to: } 0 \le x_i \le 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } \theta_i = \frac{\pi}{4(1+g(x_M))}(1+2g(x_M)x_i), \forall i = 2, 3, \ldots, (M-1)$$

$$g(x_M) = \sum_{x_i \in X_M} (x_i)^{0.1}$$

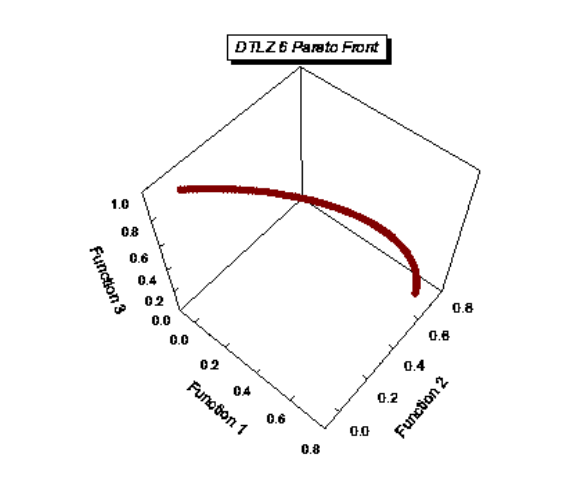
Test Problems

DTLZ6

The Pareto optimal set is located at $x_i = 0$ for every $x_i \in x_M$.

The size of the vector x_M is chosen as 10 and the total number of decision variables is identical to the one used for DTLZ5.

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ6**

DTLZ7

Minimize:

$$\begin{aligned} f_1(x) &= x_1, \\ f_2(x) &= x_2, \\ \vdots & \quad \vdots \\ f_{M-1}(x) &= x_{M-1} \\ f_M(x) &= (1 + g(x_M)) \cdot h(f_1, f_2, \ldots, f_{M-1} g(x)) \\ \text{subject to: } 0 \leq x_i \leq 1 & \quad \forall \quad i = 1, 2, ..., n \\ \text{where: } g(x) &= 1 + \frac{9}{|x_M|} \sum_{x_i \in x_M} x_i, \\ h(f_1, f_2, \ldots, f_{M-1}, g) &= M - \sum_{i=1}^{M-1} \left(\frac{f_i}{1 + g(x)} (1 + sin(3\pi f_i)) \right) \end{aligned}$$

Test Problems

DTLZ7

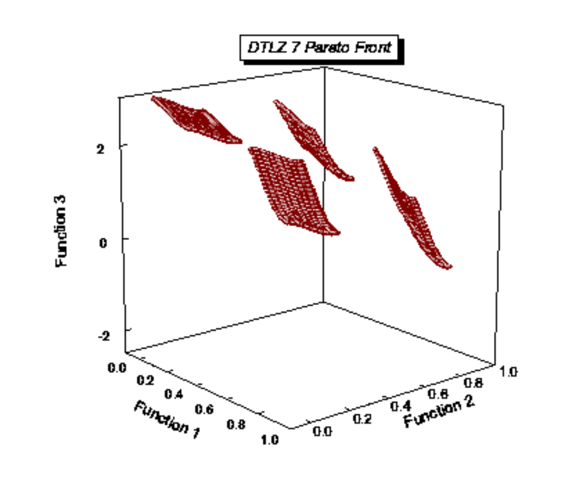
This problem has $2M - 1$ disconnected Pareto optimal regions.

g requires $k = |x_M j|$ decision variables and the total number of decision variables is $n = M + k - 1$. It is suggested to use $k = 20$.

The Pareto optimal set corresponds to $x_M = 0$.

This problem aims to test the ability of a MOEA to maintain, simultaneously, solutions at different regions of the search space.

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ7**

DTLZ8

Minimize:

$$f_j(x) = \frac{1}{\lfloor n/M \rfloor} \sum_{\lfloor i=(j-1)\frac{n}{M} \rfloor}^{\lfloor j\frac{n}{M} \rfloor} (x_i), \forall j = 1, 2, \ldots, M,$$

$$\text{subject to: } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } g_j(x) = f_M(x) + 4f_j(x) - 1 \geq 0, \forall j = 1, 2, \ldots, (M-1)$$

$$g_M(x) = 2f_M(x) + min_{i,j=1,i\neq j}^{M-1} [f_i(x) + f_j(x)] - 1 \geq 0,$$

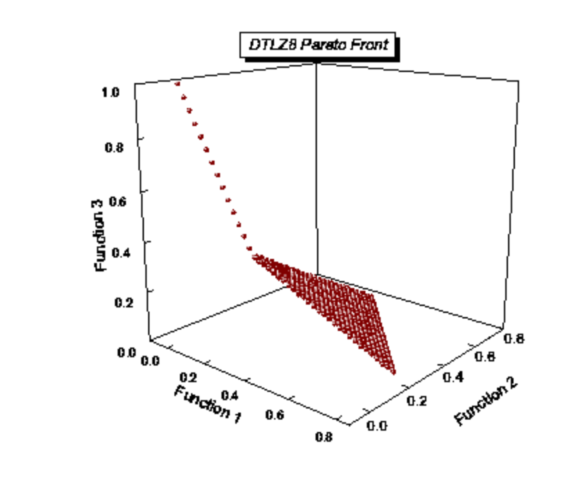
DTLZ8

The number of decision variables must be larger than the number of objectives $n > M$. It is suggested to use $n = 10M$.

This problem has M constraints. The true Pareto front is a combination of a straight line and a hyperplane.

The straight line is the intersection of the first $(M - 1)$ constraints (with $f_1 = f_2 = \ldots = f_M - 1$ and the hyperplane is represented through constraint g_M).

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ8**

Test Problems

DTLZ9

Minimize:

$$f_j(x) = \frac{1}{\lfloor n/M \rfloor} \sum_{\lfloor i=(j-1)\frac{n}{M} \rfloor}^{\lfloor j\frac{n}{M} \rfloor} \left(x_i^{0.1} \right), \forall j = 1, 2, \ldots, M,$$

$$\text{subject to: } 0 \leq x_i \leq 1 \quad \forall \quad i = 1, 2, ..., n$$

$$\text{where: } g_j(x) = f_M^2(x) + f_j^2(x) - 1 \geq 0, \forall j = 1, 2, \ldots, (M-1)$$

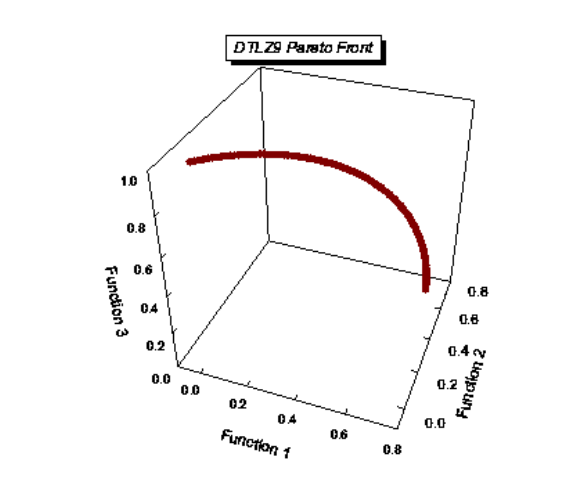
DTLZ9

The number of decision variables must be larger than the number of objectives. It is suggested to use: $n = 10M$.

The true Pareto front is a curve with $f_1 = f_2 = \ldots = f_M - 1$, similar to the Pareto front of DTLZ5. However, in this case, the density of solutions decreases as we approach the Pareto optimal region.

The Pareto front is at the intersection of all the $(M - 1)$ constraints, which can cause difficulties to a MOEA.

Test Problems



DTLZ Test Problems

PF_{true} of **DTLZ9**

Okabe's Test Problems

Tatsuya Okabe et. al [2004] proposed a methodology to generate multi-objective test problems based on a mapping of probability density functions from decision variable space to objective function space. They also provide two examples of this methodology.

The basic idea is to depart from an initial space (called S^2) between decision variable space and objective function space and from there, they build both spaces by applying appropriate functions to S^2. For this sake, the authors proposed to use the inverse of the generation operation (i.e., deformation, rotation and translation).

Tatsuya Okabe, Yaochu Jin, Markus Olhofer and Bernhard Sendhoff, "**On Test Functions for Evolutionary Multi-objective Optimization**", in Xin Yao et al. (editors), *Parallel Problem Solving from Nature - PPSN VIII*, Springer-Verlag, Lecture Notes in Computer Science, Vol. 3242, pp. 792–802, Birmingham, UK, September 2004.

Okabe's Test Problems

OKA1:

Minimize:

$$
\begin{aligned}
f_1 &= x_1', \\
f_2 &= \sqrt{2\pi} - \sqrt{|x_1'|} + 2|x_2' - 3\cos(x_1') - 3|^{\frac{1}{2}}, \\
&\text{where:} \\
x_1' &= \cos(\pi/12)x_1 - \sin(\pi/12)x_2, \\
x_2' &= \sin(\pi/12)x_1 + \cos(\pi/12)x_2, \\
&\text{subject to:} \\
x_1 &\in [6\sin(\pi/12), 6\sin(\pi/12) + 2\pi\cos(\pi/12)], \\
x_2 &\in [-2\pi\sin(\pi/12), 6\cos(\pi/12)],
\end{aligned} \tag{13}
$$

Okabe's Test Problems

The Pareto optimal set is located at: $x'_2 = 3\cos(x'_1 + 3)$ and $x'_1 \in [0, 2\pi]$.

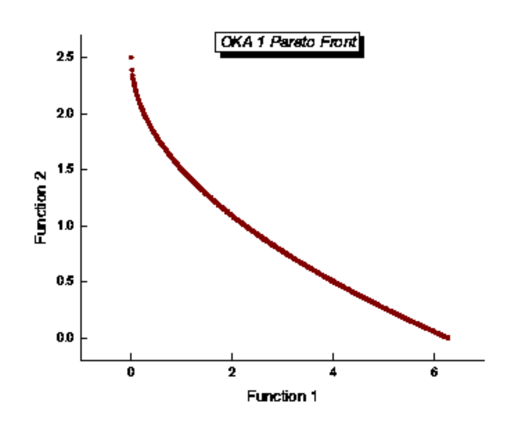
The Pareto front is located at: $f_2 = \sqrt{(2\pi)} - \sqrt{f_1}$ and $f_1 \in [-\pi, \pi]$.

The Distribution indicator is:

$$D_{x \to f} = \frac{3}{2}|x'_2 - 3\cos(x'_1) - 3|^{\frac{2}{3}} \tag{14}$$

The Distribution indicator measures the amount of distortion that the probability density suffers in decision variable space under the mapping from decision variable space to objective function space.

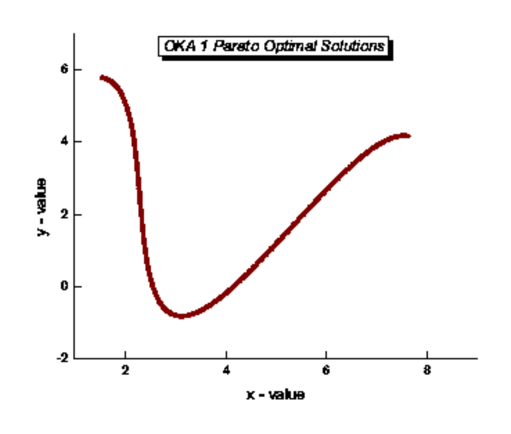
Test Problems



Okabe's Test Problems

Pareto front of OKA1

Test Problems



Okabe's Test Problems

Pareto optimal set of OKA1

Okabe's Test Problems

OKA2:

Minimize:

$$f_1 = x_1,$$

$$f_2 = 1 - \frac{1}{4\pi^2}(x_1 + \pi)^2 + |x_2 - 5\cos(x_1)|^{\frac{1}{3}} + |^{x}_{3} - 5\sin(x_1)|^{\frac{1}{3}},$$

subject to:

$$x_1 \in [-\pi, \pi],$$

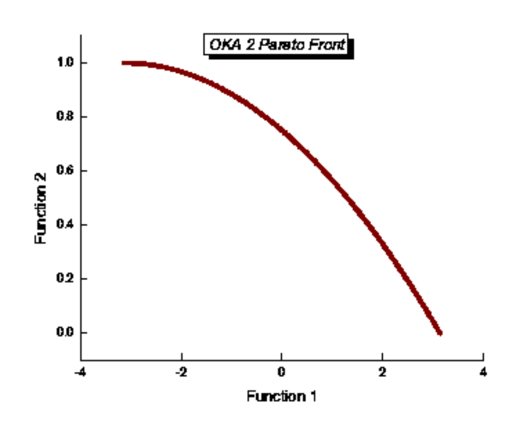
$$x_2, x_3 \in [-5, 5]$$

Okabe's Test Problems

The Pareto optimal set is located at: $(x_1, x_2, x_3) = (x_1, 5\cos(x_1), 5\sin(x_1))$ and $x_1 \in [-\pi, \pi]$.

The true Pareto front is located at: $f_2 = 1 - \frac{1}{4\pi^2}(f_1 + \pi)^2$ and $f_1 \in [-\pi, \pi]$.

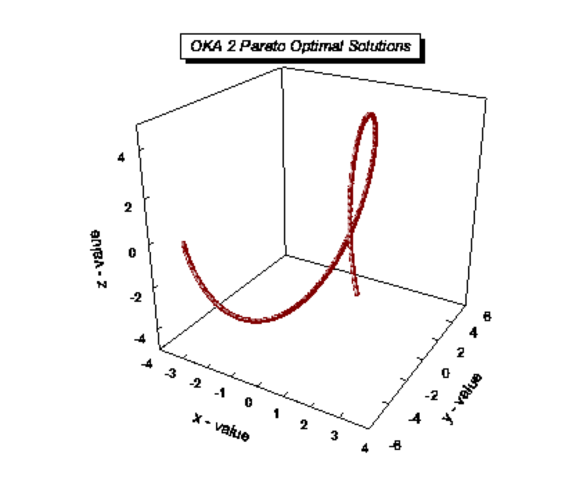
The Distribution indicator is: $D_{x \to f} = 9|x_2 - 5\cos(x_1)|^{\frac{2}{3}}|x_3 - 5\sin(x_1)|^{\frac{2}{3}}$.



Okabe's Test Problems

Pareto front of OKA2

Test Problems



Okabe's Test Problems

Pareto optimal set of OKA2

Test Problems

WFG Test Problems

Huband et al. [2006] proposed a methodology to design test problems which are quite challenging for MOEAs. The set that they used to exemplify their methodology is known as the Walking-Fish-Group (WFG) test suite.

In the next slides, we show the shapes for the objective functions and the transformation functions.

Simon Huband, Phil Hingston, Luigi Barone and Lyndon While, "**A Review of Multiobjective Test Problems and a Scalable Test Problem Toolkit**", *IEEE Transactions on Evolutionary Computation*, Vol. 10, No. 5, pp. 477–506, October 2006.

WFG Test Problems (Shapes for the objective functions)

$$\begin{aligned}
\text{linear}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1} x_i \\
\text{linear}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m} x_i\right)(1-x_{M-m+1}) \\
\text{linear}_M(x_1,\ldots,x_{M-1}) &= 1-x_1
\end{aligned}$$

$$\begin{aligned}
\text{convex}_1(x_1,\ldots,x_{M-1}) &= \prod_{i=1}^{M-1}(1-\cos(x_i\pi/2)) \\
\text{convex}_{m=2:M-1}(x_1,\ldots,x_{M-1}) &= \left(\prod_{i=1}^{M-m}(1-\cos(x_i\pi/2))\right)(1-\sin(x_{M-m+1}\pi/2)) \\
\text{convex}_M(x_1,\ldots,x_{M-1}) &= 1-\sin(x_1\pi/2)
\end{aligned}$$

WFG Test Problems (Shapes for the objective functions)

$$
\begin{aligned}
\mathrm{concave}_1(x_1, \ldots, x_{M-1}) &= \prod_{i=1}^{M-1} \sin(x_i \pi/2)) \\
\mathrm{concave}_{m=2:M-1}(x_1, \ldots, x_{M-1}) &= \left(\prod_{i=1}^{M-m} \sin(x_i \pi/2) \right) \cos(x_{M-m+1} \pi/2) \\
\mathrm{concave}_M(x_1, \ldots, x_{M-1}) &= \cos(x_1 \pi/2)
\end{aligned}
$$

$$
\mathrm{mixed}_M(x_1, \ldots, x_{M-1}) = \left(1 - x_1 - \frac{\cos(2A\pi x_1 + \pi/2)}{2A\pi} \right)^{\alpha}
$$

$$
\mathrm{disc}_M(x_1, \ldots, x_{M-1}) = 1 - x_1^{\alpha} \cos^2(A x_1^{\beta} \pi)
$$

WFG Test Problems (Transformation functions)

$$\text{b\_poly}(y, \alpha) = y^{\alpha}$$

$$\text{b\_flat}(y, A, B, C) = A + \min(0, \lfloor y - B \rfloor)\frac{A(B-y)}{B} - \min(0, \lfloor C - y \rfloor)\frac{(1-A)(y-C)}{1-C}$$

$$\text{b\_param}(y, u(\vec{y}'), A, B, C) = y^{B+(C-B)\left(A-(1-2u(\vec{y}'))\left|\lfloor 0.5-u(\vec{y}') \rfloor + A\right|\right)}$$

$$\text{s\_linear}(y, A) = \frac{|y-A|}{|\lfloor A - y \rfloor + A|}$$

$$\text{s\_decept}(y, A, B, C) = 1+(|y-A|-B)\left(\frac{\lfloor y-A+B \rfloor\left(1-C+\frac{A-B}{B}\right)}{A-B} + \frac{\lfloor A+B-y \rfloor\left(1-C+\frac{1-A-B}{B}\right)}{1-A-B} + \frac{1}{B}\right)$$

$$\text{s\_multi}(y, A, B, C) = \frac{1+\cos\left((4A+2)\pi\left(0.5-\frac{|y-C|}{2(\lfloor C-y \rfloor + C)}\right)\right) + 4B\left(\frac{|y-C|}{2(\lfloor C-y \rfloor + C)}\right)^2}{b+2}$$

$$\text{r\_sum}(\vec{y}, \vec{w}) = \frac{\sum_{i=1}^{|\vec{y}|} w_1 y_i}{\sum_{i=1}^{|\vec{y}|} w_i}$$

$$\text{r\_nonsep}(\vec{y}, A) = \frac{\sum_{j=1}^{|\vec{y}|}\left(y_j + \sum_{k=0}^{A-2}\left|y_j - y_{1+(j+k)mod|\vec{y}|}\right|\right)}{\frac{|\vec{y}|}{A}\left\lceil \frac{A}{2} \right\rceil\left(1+2A-2\left\lceil \frac{A}{2} \right\rceil\right)}$$

WFG1:
Minimize

$$
\begin{aligned}
f_{m=1:M-1}(\vec{x}) &= x_M + S_m \text{convex}_m(x_1, \ldots, x_{M-1}) \\
f_M(\vec{x}) &= x_M + S_M \text{mixed}_M(x_1, \ldots, x_{M-1})
\end{aligned}
$$

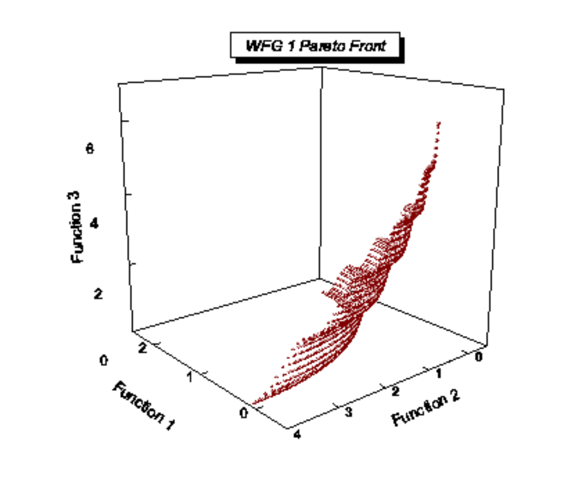
where

$$
\begin{aligned}
y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [2((i-1)k/(M-1)+1), \ldots, 2ik/(M-1)]) \\
y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [2(k+1), \ldots, 2n]) \\
y'_{i=1:n} &= \text{b\_poly}(y''_i, 0.02) \\
y''_{i=1:k} &= y'''_i \\
y''_{i=k+1:n} &= \text{b\_flat}(y'''_i, 0.8, 0.75, 0.85) \\
y'''_{i=1:k} &= z_{i,[0,1]} \\
y'''_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35)
\end{aligned}
$$

Test Problems

For all problems:
The decision vector is $z = [z_1, \ldots, z_k, z_{k+1}, \ldots, z_n]$ where $0 \leq z_i \leq z_{i,\max}$.

$$
\begin{aligned}
z_{i=1:n,\max} &= 2i \\
z_{i=1:n,[0,1]} &= \frac{z_i}{z_{i,\max}} \\
x_{i=1:M-1} &= \max(y_M, A_i)(y_i - 0.5) + 0.5 \\
x_M &= y_M \\
S_{m=1:M} &= 2m \\
A_1 &= 1 \\
A_{2:M-1} &= \begin{cases} 0, & \text{for WFG3} \\ 1, & \text{otherwise} \end{cases}
\end{aligned}
$$



WFG Test Problems

Pareto front of WFG1

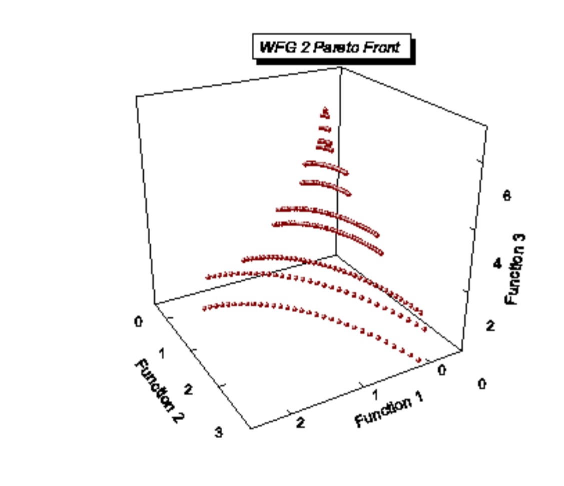
WFG2:
Minimize

$$f_{m=1:M-1}(\vec{x}) = x_M + S_m \mathrm{convex}_m(x_1, \ldots, x_{M-1})$$
$$f_M(\vec{x}) = x_M + S_M \mathrm{disc}_M(x_1, \ldots, x_{M-1})$$

where

$$y_{i=1:M-1} = \mathrm{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1])$$
$$y_M = \mathrm{r\_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1])$$
$$y'_{i=1:k} = y''_i$$
$$y'_{i=k+1:k+l/2} = \mathrm{r\_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2)$$
$$y''_{i=1:k} = z_{i,[0,1]}$$
$$y''_{i=k+1:n} = \mathrm{s\_linear}(z_{i,[0,1]}, 0.35)$$

Test Problems



WFG Test Problems

Pareto front of WFG2

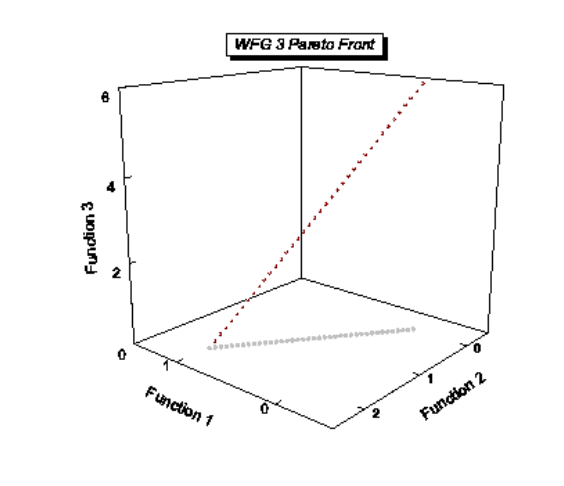
WFG3:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{linear}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \text{r_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r_sum}([y'_{k+1}, \ldots, y'_{k+l/2}], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:k+l/2} &= \text{r_nonsep}([y''_{k+2(i-k)-1}, y''_{k+2(i-k)}], 2) \\
y''_{i=1:k} &= z_{i,[0,1]} \\
y''_{i=k+1:n} &= \text{s_linear}(z_{i,[0,1]}, 0.35)
\end{aligned}$$

Test Problems



WFG Test Problems

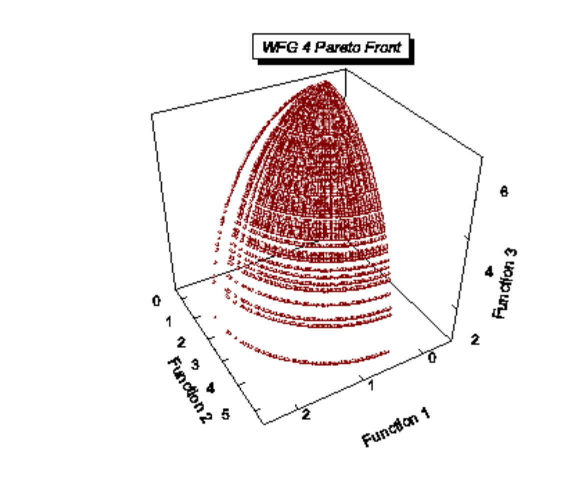
Pareto front of WFG3

WFG4:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\ y'_{i=1:n} &= \text{s\_multi}(z_{i,[0,1]}, 30, 10, 0.35) \end{aligned}$$



WFG Test Problems

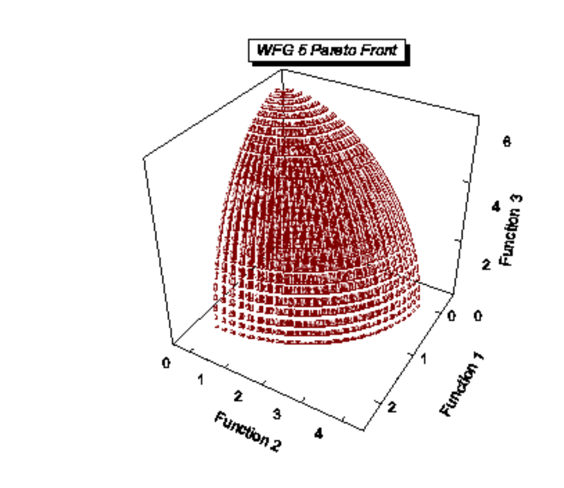
Pareto front of WFG4

WFG5:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\ y_M &= \text{r\_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\ y'_{i=1:n} &= \text{s\_decept}(z_{i,[0,1]}, 0.35, 0.001, 0.05) \end{aligned}$$



WFG Test Problems

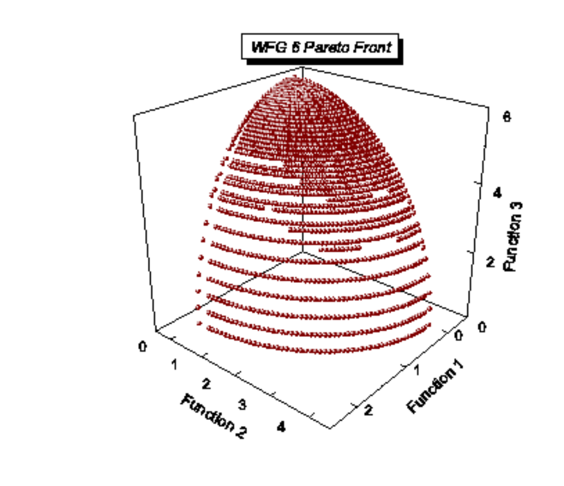
Pareto front of WFG5

WFG6:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned} y_{i=1:M-1} &= \text{r\_nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\ y_M &= \text{r\_nonsep}([y'_{k+1}, \ldots, y'_n], l) \\ y'_{i=1:k} &= z_{i,[0,1]} \\ y'_{i=k+1:n} &= \text{s\_linear}(z_{i,[0,1]}, 0.35) \end{aligned}$$



WFG Test Problems

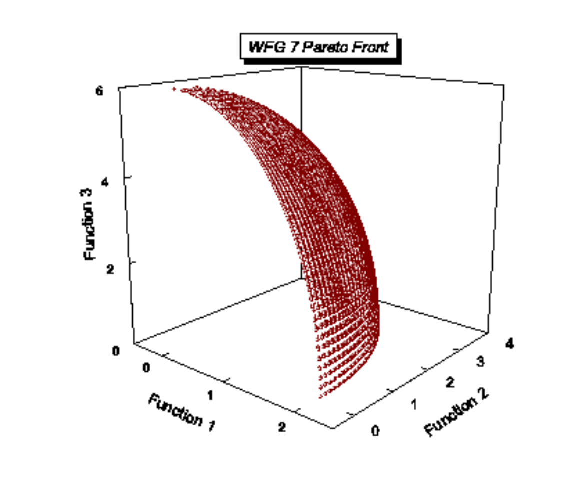
Pareto front of WFG6

WFG7:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \text{r_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:n} &= \text{s_linear}(y''_i, 0.35) \\
y''_{i=1:k} &= \text{b_param}(z_{i,[0,1]}, \text{r_sum}([z_{i+1,[0,1]}, \ldots, z_{n,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_{i=k+1:n} &= z_{i,[0,1]}
\end{aligned}$$



WFG Test Problems

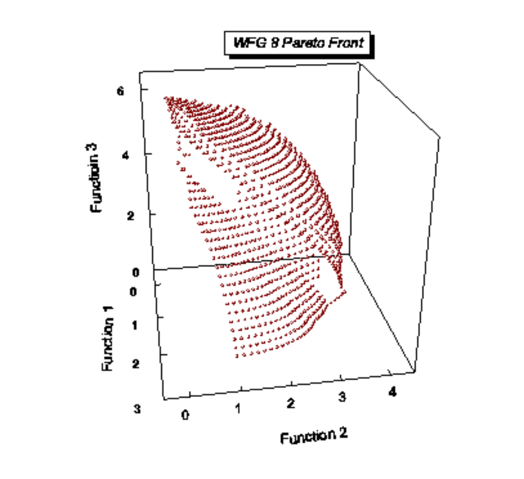
Pareto front of WFG7

WFG8:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \text{r_sum}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], [1, \ldots, 1]) \\
y_M &= \text{r_sum}([y'_{k+1}, \ldots, y'_n], [1, \ldots, 1]) \\
y'_{i=1:k} &= y''_i \\
y'_{i=k+1:n} &= \text{s_linear}(y''_i, 0.35) \\
y''_{i=1:k} &= z_{i,[0,1]} \\
y''_{i=k+1:n} &= \text{b_param}(z_{i,[0,1]}, \text{r_sum}([z_{1,[0,1]}, \ldots, z_{i-1,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50)
\end{aligned}$$



WFG Test Problems

Pareto front of WFG8

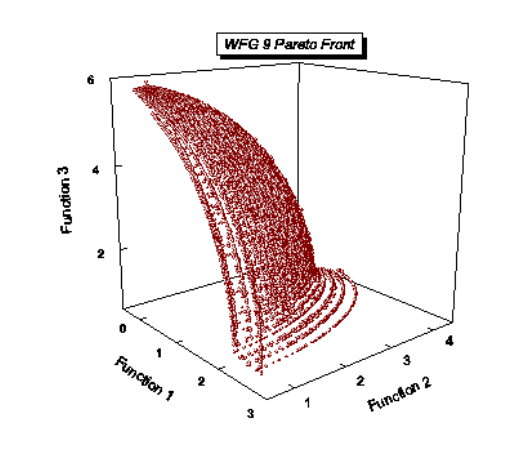
WFG9:
Minimize

$$f_{m=1:M}(\vec{x}) = x_M + S_m \text{concave}_m(x_1, \ldots, x_{M-1})$$

where

$$\begin{aligned}
y_{i=1:M-1} &= \text{r_nonsep}([y'_{(i-1)k/(M-1)+1}, \ldots, y'_{ik/(M-1)}], k/(M-1)) \\
y_M &= \text{r_nonsep}([y'_{k+1}, \ldots, y'_n], l) \\
y'_{i=1:k} &= \text{s_decept}(y''_i, 0.35, 0.001, 0.05) \\
y'_{i=k+1:n} &= \text{s_multi}(y''_i, 30, 95, 0.35) \\
y''_{i=1:n-1} &= \text{b_param}(z_{i,[0,1]}, \text{r_sum}([z_{i+1,[0,1]}, \ldots, z_{n,[0,1]}], [1, \ldots, 1]), 0.98/49.98, 0.02, 50) \\
y''_n &= z_{n,[0,1]}
\end{aligned}$$

Test Problems



WFG Test Problems

Pareto front of WFG9

Test Problems

Use of Lamé Superspheres

In order to design multi-objective test problems with different Pareto optimal fronts, Emmerich and Deutz [2007] introduced a scalable test suite based on the **Lamé superspheres** (LSS). Although this methodology is limited to design Pareto optimal geometries with spherical shapes, it can be considered as the first study focused on the Pareto shape of multi-objective test problems. Something remarkable about this proposal are the mirror test problems which adopt an inverted sphere as the Pareto shape of the proposed multi-objective test problems. Even though the use of mirror spheres had already been adopted as a Pareto optimal surface by Huband et al. [2006], the parameter γ of the Lamé spheres is able to modify the convexity/concavity degree in these test problems.

Michael T.M. Emmerich and André H. Deutz, "**Test Problems Based on Lamé Superspheres**", in Shigeru Obayashi et al. (Eds), *Evolutionary Multi-Criterion Optimization, 4th International Conference, EMO 2007*, pp. 922–936, Springer. Lecture Notes in Computer Science Vol. 4403, Matshushima, Japan, March 2007.

Use of Lamé Superspheres

In addition to new Pareto optimal shapes, the Lamé superspheres test suite incorporates features such as multi-modality and many-to-one mapping which introduce additional difficulties for solving these problems using a MOEA. Since this test suite adopts the DTLZ framework, distance and position parameters can also be easily identified.

Test Problems

Complicated Pareto Sets

Saxena et al. [2011] extended the principle of complicated Pareto sets (PSs) initially introduced for two- and three-objective problems [Zhang, 2009] to scalable multi-objective test problems. The **Saxena-Zhang-Duro-Tieari** (SZDT) test suite introduces seven unconstrained test problems and the possibility of designing new test problems by choosing a combination between Pareto optimal shapes and complicated PS topologies.

Dhish Kumar Saxena, Qingfu Zhang, João A. Duro and Ashutosh Tiwari, "**Framework for Many-Objective Test Problems with Both Simple and Complicated Pareto-Set Shapes**", in Ricardo H.C. Takahashi et al. (Eds), *Evolutionary Multi-Criterion Optimization, 6th International Conference, EMO 2011*, pp. 197–211, Springer. Lecture Notes in Computer Science Vol. 6576, Ouro Preto, Brazil, April 2011.

Hui Li and Qingfu Zhang, "**Multiobjective Optimization Problems with Complicated Pareto Sets, MOEA/D and NSGA-II**", *IEEE Transactions on Evolutionary Computation*, Vol. 13, No. 2, pp. 284–302, April 2009.

Complicated Pareto Sets

The Pareto optimal fronts for all these test problems are defined in one and two dimensions, i.e., they become degenerate for more than two and three objectives, respectively.

Regarding the Pareto optimal fronts, four continuous and connected surfaces including convexity and concavity generalize the Pareto shapes in this test suite. The convergence difficulties in this benchmark are specifically stated by the topology of the PSs.

The absence of multi-modality and non-separability, are the shortcomings in this test suite. However, the use of the modular approach in this testbed, makes difficult to determine position and distance parameters, which becomes an advantage over the previous test suites.

Large Scale MOPs

As pointed out by Huband et al. [2006], variable linkages should be considered in the construction of multi-objective test problems. This feature in test instances is particularly important because, it makes it more difficult for a MOEA to properly exploit optimal solutions.

Cheng et al. [2017] introduced a set of nine test problems specially designed to test MOEAs for large scale optimization (i.e., for multi-objective problems with a large number of decision variables).

Ran Cheng, Yaochu Jin, Markus Olhofer and Bernhard Sendhoff, "**Test Problems for Large-Scale Multiobjective and Many-Objective Optimization**", *IEEE Transactions on Cybernetics*, Vol. 47, No. 12, pp. 4108–4121, December 2017.

Large Scale MOPs

In the **Large Scale Multi-Objective Problems** (LSMOPs), the variable dependencies are stated by two linear variable linkage functions (linear and nonlinear).

In addition to the dependencies among variables, this test suite introduces correlation between decision variables and objectives by means of a correlation matrix. Although the test problems are scalable to an arbitrary number of objectives, this test suite is limited to three Pareto optimal shapes, concretely, the PFs from DTLZ1 (normalized in objective function space), DTLZ2, and DTLZ7.

Test Problems

A Toolkit

Masuda et al. [2016] proposed a toolkit to generate scalable test problems. This test suite is mainly focused on the design of different Pareto optimal shapes. The methodology introduced in this approach allows the design of Pareto optimal surfaces by using a finite number of vertices. Such vertices state the Pareto optimal front whose shape can be defined as linear, concave, or convex. Although only two test problems were instantiated, the toolkit provides a methodology for designing scalable test problems with Pareto optimal surfaces having an arbitrary number of vertices.

Hiroyuki Masuda, Yusuke Nojima and Hisao Ishibuchi, "**Common Properties of Scalable Multiobjective Problems and a New Framework of Test Problems**", in *2016 IEEE Congress on Evolutionary Computation (CEC'2016)*, pp. 3011–3018, IEEE Press, Vancouver, Canada, 24–29 July 2016, ISBN 978-1-4799-1488-3.

A Toolkit

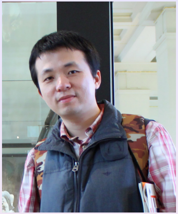
A remarkable aspect of the **Masuda-Nojima-Ishibuchi** (MNI) test suite, is that at different distances from the PF, different PF shapes can be produced. This offers certain difficulty in identifying position and distance parameters.

Multi-Line Distance Minimization Problems

Li et al. [2018] proposed and discussed a class of scalable multi-objective test instances called multi-line distance minimization problems (ML-DMP) to evaluate the performance of evolutionary approaches in high-dimensional objective spaces. The test problems proposed in this test suite, were mainly introduced for visual examination of solution diversity in the decision space instead of the objective space.

Miqing Li, Crinan Grosan, Shengxiang Yang, Xiaohui Liu and Xin Yao, "**Multi-Line Distance Minimization: A Visualized Many-Objective Test Problem Suite**", *IEEE Transactions on Evolutionary Computation*, Vol. 22, No. 1, pp. 61–78, February 2018.

Test Problems



Multi-Line Distance Minimization Problems

Two are the main characteristics of this test suite: 1) the Pareto optimal solutions lie in a regular polygon in a two-dimensional decision space, and 2) these solutions are similar (in the sense of Euclidean geometry) to their images in high-dimensional spaces. This allows to understand the distribution of the objective vector set by observing the solution set in the two-dimensional decision space in which these test problems are defined.

Minus Problems

Ishibuchi et al. [2017] proposed minus versions of the DTLZ and WFG test problems (namely minus-DTLZ ($DTLZ^{-1}$) and minus-WFG (WFG^{-1}), respectively) as scalable test problems with clear differences from their original versions. These test problems stand out mainly because the Pareto optimal fronts of the original DTLZ and WFG test problems are inverted to obtain a similar effect as in the mirror LSS test problems [Emmerich, 2007].

Hisao Ishibuchi, Yu Setoguchi, Hiroyuki Masuda and Yusuke Nojima, "**Performance of Decomposition-Based Many-Objective Algorithms Strongly Depends on Pareto Front Shapes**", *IEEE Transactions on Evolutionary Computation*, Vol. 21, No. 2, pp. 169–190, April 2017.

Minus Problems

However, in this test suite, different geometries (the geometries used in the DTLZ and WFG test problems) are employed instead of being limited to the superspheres as in the case of the mirror LSS test problems.

Some important key points to consider are the following: 1) all the test problems maintain the same properties respect to the difficulties of the distance functions; and 2) different test problems promote the design of diversity mechanisms to achieve a proper representation of the inverted DTLZ and WFG Pareto optimal fronts.

MaF

Cheng et al. [2017] presented a compilation of 15 test problems which are presented as a scalable test suite, called MaF. In this test suite, the authors' intention is to compile a set of test problems with different features in order to evaluate many-objective evolutionary approaches.

Most of the test problems included in this test suite were taken from already formulated test problems such as WFG, DTLZ, and ML-DMP, among other test suites. Thus, a wide variety of features can be found in this test suite which, indeed, shall be able to assess the robustness of many-objective evolutionary approaches.

Ran Cheng, Miqing Li, Ye Tian, Xingyi Zhang, Shengxiang Yang, Yaochu Jin and Xin Yao, "**A benchmark test suite for evolutionary many-objective optimization** ", *Complex & Intelligence Systems*, Vol. 3, No. 1, pp. 67–81, March 2017.

Test Problems

Approaches to Design MOPs

In general, there are three different techniques which have been adopted in the construction of multi-objective test problems [Deb et al., 2005]: (1) Multiple single-objective approach, (2) Bottom-Up approach, and (3) Constraint surface approach.

Multiple Single-Objective Approach

This is an intuitive method that combines a number of single-objective optimization problems to formulate a multi-objective model. This strategy was extensively adopted in the early days of evolutionary multi-objective optimization research. The main disadvantage of this approach is that the *Pareto set* (PS) and the *Pareto front* (PF) are unknown, and depending of the single-objective functions, they can be very difficult to state. This, in fact, complicates the analysis of results and the comparison of MOEAs may become unfair. Nonetheless, this methodology has been recently adopted to formulate new multi-objective test problems.

Bottom-Up Approach

The bottom-up approach [Deb et al., 2005] is a flexible method that has facilitated the design of multi-objective test problems. In this approach, the *Pareto optimal front*, the *objective space* and the *decision space* are separately constructed. Concretely, the decision variables are splitted into two groups: “*position*” and “*distance*” parameters.

The Pareto optimal surface is constructed by parametric functions (*position functions*) whose inputs are the position parameters. The objective space is stated by constructing an extreme boundary surface parallel to the Pareto optimal surface, so that the hyper-volume bounded by these two surfaces constitutes the attainable objective space. Finally, each decision variable vector is mapped into objective space.

Bottom-Up Approach

This task is carried out by defining linear/nonlinear functions where the inputs are the distance parameters. Such functions (known as *distance functions*) establish the distance of the objective vectors to the PF. Therefore, the difficulty to approximate solutions to the PF depends directly on the difficulty of solving such distance functions.

Because of its flexibility, the bottom-up approach has been successfully employed in the construction of multi-objective test problems, particularly in the design of scalable test problems. However, most of the test suites adopting the bottom-up approach assume that position and distance parameters are completely uncorrelated—i.e. they can be easily identified—which is something hardly seen in real-world problems.

Test Problems

Constraint Surface Approach

This method was introduced to construct constrained multi-objective test problems [Deb et al., 2005]. Unlike the bottom-up approach that starts from a pre-defined Pareto optimal surface, the constraint surface approach first states the overall search space.

Second, a number of linear/non-linear constraints involving the objective function values is added, thus erasing part of the objective space (i.e., restricting the search space). Finally, by defining linear/non-linear objective functions, the decision variable space is mapped into the objective space.

Kalyanmoy Deb, Lothar Thiele, Marco Laumanns and Eckart Zitzler, “**Scalable Test Problems for Evolutionary Multiobjective Optimization**”, in Ajith Abraham, Lakhmi Jain and Robert Goldberg (Editors), *Evolutionary Multiobjective Optimization. Theoretical Advances and Applications*, pp. 105–145, Springer, USA, 2005.

Test Problems

Recommendations and Features

The construction of multi-objective test problems should satisfy some requirements and should include characteristics aimed to evaluate specific components of MOEAs. In particular, when a test instance possesses different characteristics, the test problem should evaluate the robustness of a MOEA, i.e., the capability of a MOEA to solve a test problem with a certain number of features.

Several criteria for the construction of multi-objective test instances have been discussed by a number of researchers, particularly in the pioneering works of Deb et al. [2005] and Huband et al. [2006].

Test Problems

Recommendations and Features

Huband et al. [2006] analyzed and justified different requirements which should be considered in the design of multi-objective test problems. We will show next the seven recommendations (R1–R7) and the five features (F1–F5) discussed by Huband et al. [2006]. However, because of the inherent progress on evolutionary multi-objective optimization, other features (F6–F8) are also added and described.

Test Problems

Recommendations

R1: No Extremal Parameters	Prevents exploitation by truncation based correction operators
R2: No Medial Parameters	Prevents exploitation by intermediate recombination
R3: Scalable Number of Parameters	Increases flexibility, demands scalability
R4: Scalable Number of Objectives	Increases flexibility, demands scalability
R5: Dissimilar Parameter Domains	Encourages EAs to scale mutation strengths appropriately
R6: Dissimilar Trade-off Ranges	Encourages normalization of objective values
R7: Pareto Optima Known	Facilitates the use of measures, analysis of results, in addition to other benefits

Features

F1: Pareto Optimal Geometry	Convex, linear, concave, mixed, degenerate, disconnected, or some combination
F2: Parameter dependencies	Objectives can be separable or non-separable
F3: Bias	Substantially more solutions exist in some regions of fitness space than they do in others
F4: Many-to-one mappings	Pareto one-to-one/many-to-one, flat regions, isolated optima
F5: Modality	Uni-modal, or multi-modal (possibly deceptive multi-modality)
F6: Difficult Pareto Set Topology	Pareto set difficult to characterize
F7: Difficult Pareto Front Shape	Pareto optimal front difficult to estimate
F8: Correlation of Position and Distance Functions	Dependencies between position and distance functions
F9: Single Optimal Solution for a High Number of Objectives	Single objective solution for multiple objective functions

Test Problems

Limitations of Current Benchmarks

All modern benchmarks follow the bottom-up approach. This form to formulate multi-objective test problems splits the construction of the PF and the design of the search space which, in fact, facilitates the construction of multi-objective problems specially in high-dimensional objective spaces.

Recommendations (R1–R7) are partially covered by most of the modern test problems, being the WFG test suite, the only set of problems that satisfies entirely such requirements.

In the case of features related to the search space (F2–F5), most of the modern test problems do not adhere or cannot fit in a specific or desirable combination of features. While these features can be studied separately, there is no reason to assume that a real-world problem does not adhere simultaneously to several of these features at the same time.

Limitations of Current Benchmarks

Although one might doubt the existence of multi-objective problems having a combination of characteristics different from the ones formulated in the existing scalable test suites, according to the *No-Free Lunch* theorem, this overestimation does not hold.

In other words, there is an immense number of formulated and unformulated real-world problems and it is reasonable to think that any of them may have a wide variety of features not contemplated in any already formulated artificial test problem.

Thus, the inflexibility of configuring (in an easy way) scalable test problems with a desirable combination of features, becomes also a limitation of the existing scalable test suites.

On the other hand, difficult PS topologies (F6) are not considered by most of the modern test suites, which becomes a limitation.

Test Problems

Limitations of Current Benchmarks

An important issue to consider in scalable test problems refers to the shape of the Pareto optimal front. In this regard, the Pareto optimal fronts of the existing scalable test problems combine a variety of different geometries including convexity, concavity and/or linearity.

Several of the existing test problems (e.g., from the DTLZ and WFG test suites) can be characterized by an $(M - 1)$-simplex. Test problems having this type of shapes are easy to solve for some evolutionary approaches.

In the specialized literature, we can find several MOPs in which their PF approximations draw strange geometries that do not follow exactly the shape of an $(M - 1)$-simplex, see for example the problems presented in [Dirkx and Mooij, 2014].

D. Dirkx and E. Mooij, "**Optimization of entry-vehicle shapes during conceptual design**", *Acta Astronautica*, Vol. 94, No. 1, pp. 198–214, 2014.

Limitations of Current Benchmarks

Most of the modern test problems do not follow the property of difficult PF shape (F7) that has been suggested to evaluate diversity mechanisms in MOEAs. This, in fact, becomes a limitation of the constructed test problems and motivates to design new geometries different from those included in the state-of-the-art test suites.

Another important property that should be considered in the construction of scalable test problems is regarding the correlation between position and distance functions (F8). Most of the modern test problems do not follow this property which complicates the identification of position and distance parameters.

Although there exist approaches employed to correlate position and distance functions (e.g. the modular approach), the investigation and development of a more flexible design approach for constructing scalable test problems—where position and distance variables are indistinguishable and the true PS and PF can be analytically known—is in fact a good path for future research.